MASSEY UNIVERSITY
Institute of Fundamental Sciences
Mathematics

160.734 Studies in Applied Differential Equations

Semester Test

Semester Two — September 2014

Time allowed: **55 minutes**

This is a **closed book** examination.

Total marks: 40

Attempt all questions. There are 6 questions altogether.
Be sure to read each question carefully.

Show all working for full credit.

1. Let $A = \begin{bmatrix} 8 & 8 \\ 0 & 8 \end{bmatrix}$.

   (a) Compute e^{tA} (where $t \in \mathbb{R}$).

   (b) Use your answer from (a) to state the solution to $\dot{x}(t) = Ax(t)$ with an arbitrary initial condition, $x(0) = x_0$.

[6 + 1 = 7 marks]

2. Let $B = \begin{bmatrix} -1 & 0 & 0 \\ -2 & 4 & 0 \\ 2 & 5 & -6 \end{bmatrix}$, and consider the ODE, $\dot{x}(t) = Bx(t)$, with $x(0) = x_0$.
   Describe all $x_0 \in \mathbb{R}^3$ for which $x(t) \to 0$ as $t \to \infty$.

[6 marks]

3. Consider the ODE, $\dot{x}(t) = f(x(t))$, with $x(0) = 0$.
Give a example of a *continuous* function $f : \mathbb{R} \to \mathbb{R}$ for which more than one value is possible for $x(1)$.
Provide a brief explanation or calculation to justify your answer.
Hint: f cannot be Lipschitz.

[5 marks]

4. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 function, and let x^* be an equilibrium of $\dot{x}(t) = f(x(t))$.

   (a) Define what it means for x^* to be *hyperbolic*.

   (b) Define what it means for x^* to be *Lyapunov stable*.

[3 + 3 = 6 marks]

5. Consider the system

$$\dot{x} = 4 - y + x^2 \ ,$$
$$\dot{y} = 1 + 4x - y \ .$$

   (a) Find all equilibria.

   (b) Classify each equilibrium (as either a stable node, a stable focus, an unstable node, an unstable focus, a saddle, or none of the above).

[3 + 5 = 8 marks]

6. Consider the system

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = f(x, y) = \begin{bmatrix} -3x + y \\ 5xy - 4y^2 \end{bmatrix} \ .$$

The origin $(0, 0)$ is an equilibrium, and the eigenvalues of $Df(0, 0)$ are 0 and -3.

   (a) The centre manifold $W^c(0, 0)$ is given by $y = 3x + \alpha x^2 + \mathcal{O}(x^3)$, for some $\alpha \in \mathbb{R}$.
   Determine the value of α.

   (b) On $W^c(0, 0)$, we have $\dot{x} = \gamma x^2 + \mathcal{O}(x^3)$, for some $\gamma \in \mathbb{R}$.
   Determine the value of γ.

   (c) Is $(0, 0)$ asymptotically stable? Explain why or why not.

[4 + 2 + 2 = 8 marks]

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