

MASSEY UNIVERSITY
Institute of Fundamental Sciences
Mathematics

160.734 Studies in Applied Differential Equations

Semester Test

Semester One — April 2015

Time allowed: **55 minutes**

This is a **closed book** examination.

Total marks: 40

Attempt all questions. There are 5 questions altogether.
Be sure to read each question carefully.

Show all working for full credit.

1. Let $A = \begin{bmatrix} -4 & 3 \\ 2 & 1 \end{bmatrix}$, and consider the IVP

$$\dot{x} = Ax, \quad x(0) = x_0.$$

Describe all $x_0 \in \mathbb{R}^2$ for which $|x(t)| \rightarrow \infty$ as $t \rightarrow \infty$.

[8 marks]

2. Let A and B be similar $n \times n$ matrices.

Let $\varphi_t(x_0)$ and $\psi_t(x_0)$ be the flows of $\dot{x} = Ax$ and $\dot{x} = Bx$, respectively.

Prove that φ and ψ are *diffeomorphic*, that is, there exists a diffeomorphism h such that

$$h(\varphi_t(x_0)) = \psi_t(h(x_0)),$$

for all $x_0 \in \mathbb{R}^2$ and $t \in \mathbb{R}$.

[8 marks]

3. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a C^1 function, and let x^* be an equilibrium of $\dot{x} = f(x)$.

(a) Define what it means for x^* to be *hyperbolic*.

(b) Define $W^s(x^*)$, the *stable manifold* of x^* .

[3 + 3 = 6 marks]

4. Prove that the origin is an asymptotically stable equilibrium of the system

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{cases} \begin{bmatrix} y \\ -x \end{bmatrix}, & x < 0 \\ \begin{bmatrix} -x + y \\ -x - y \end{bmatrix}, & x > 0 \end{cases}.$$

[8 marks]

5. For the system

$$\begin{aligned} \dot{x} &= x - 2y, \\ \dot{y} &= 1 - e^x, \end{aligned}$$

the unstable manifold of the origin, $W^u(0, 0)$, may be written in the form

$$y = ax + bx^2 + \mathcal{O}(x^3).$$

Determine the values of a and b .

[10 marks]

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