

MASSEY UNIVERSITY
Institute of Fundamental Sciences
Mathematics

160.734 Studies in Applied Differential Equations

Semester Test

Semester One — May 2016

Time allowed: **55 minutes**

This is a **closed book** examination.

Total marks: 40

Attempt all questions. There are 5 questions altogether.
Be sure to read each question carefully.

Show all working for full credit.

1. Let $f(x) = x^3 + \frac{1}{x}$ and $g(x) = 2x^3$.

(a) Show that $g(x) \geq f(x)$ for all $x \geq 1$.

(b) Solve the IVP

$$\dot{x} = g(x), \quad x(0) = 1.$$

(c) There exists $T > 0$ such that the IVP

$$\dot{x} = f(x), \quad x(0) = 1,$$

has a solution for all $t \in [0, T)$ (do not attempt to compute this solution).
Use the results of (a) and (b) to identify a lower bound for value of T .

[2 + 4 + 4 = 10 marks]

2. Let $A = \begin{bmatrix} -1 & 5 \\ -4 & 3 \end{bmatrix}$.

(a) Compute e^{tA} (where $t \in \mathbb{R}$).

(b) Use your answer to (a) to state the solution to

$$\dot{x} = Ax, \quad x(0) = x_0$$

where $x_0 \in \mathbb{R}^2$.

[6 + 2 = 8 marks]

3. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a continuous function.

(a) Define what it means for $x^* \in \mathbb{R}^n$ to be an *equilibrium* of $\dot{x} = f(x)$.

(b) Define what it means for x^* to be *Lyapunov stable*.

[2 + 3 = 5 marks]

4. Consider the system

$$\begin{aligned} \dot{x} &= x - y - 1, \\ \dot{y} &= 4x^2 + 2x + y. \end{aligned}$$

(a) Find all equilibria.

(b) Classify each equilibrium as either a stable node, a stable focus, an unstable node, an unstable focus, a saddle, or none of the above.

[4 + 4 = 8 marks]

5. Consider the system

$$\begin{aligned} \dot{x} &= 4x + y + 2xy, \\ \dot{y} &= -xy. \end{aligned}$$

(a) Calculate $W^c(0,0)$ to third order.

(b) Derive an ODE for the dynamics on $W^c(0,0)$ to third order.

[6 + 3 = 9 marks]

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