MASSEY UNIVERSITY
Institute of Fundamental Sciences
Mathematics

160.734 Studies in Applied Differential Equations

Semester Test

Semester One — May 2017

Time allowed: **55 minutes**

This is a **closed book** examination.

Total marks: 40

Attempt all questions. There are 6 questions altogether.
Be sure to read each question carefully.

Show all working for full credit.

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1. Compute e^{tA} where $A = \begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix}$. HINT: A is nilpotent.

[5 marks]

We have $A^2 = \begin{bmatrix} 0 & 0 & 15 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $A^k = 0$ for all $k \geq 3$.

Thus

$$\begin{aligned} \mathrm{e}^{tA} &= I + tA + \tfrac{1}{2}t^2 A^2 \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + t \begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix} + \frac{t^2}{2} \begin{bmatrix} 0 & 0 & 15 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 3t & 4t + \frac{15}{2}t^2 \\ 0 & 1 & 5t \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

2. Consider the linear system

$$\dot{x} = \alpha x - y \ ,$$
$$\dot{y} = (\alpha + 1)x \ .$$

Determine the values of $\alpha \in \mathbb{R}$ for which the equilibrium $(x, y) = (0, 0)$ is Lyapunov stable but not asymptotically stable.

[6 marks]

The Jacobian of the origin is $A = \begin{bmatrix} \alpha & -1 \\ \alpha + 1 & 0 \end{bmatrix}$.
The trace and determinant of A are $\tau = \alpha$ and $\delta = \alpha + 1$, respectively.
The origin is Lyapunov stable but not asymptotically stable if either (i) $\tau = 0$ and $\delta > 0$, which occurs for $\alpha = 0$, or (ii) $\tau < 0$ and $\delta = 0$, which occurs for $\alpha = -1$.

3. Consider the system

$$\dot{x} = (x + 1)^2 - y \ ,$$
$$\dot{y} = x - y + 3 \ .$$

(a) Find all equilibria of the system.

(b) Classify each equilibrium as a stable node, a stable focus, an unstable node, an unstable focus, a saddle, or none of the above.

[3 + 6 = 9 marks]

(a) Solving $\dot{x} = 0$ and $\dot{y} = 0$ simultaneously gives

$$y = (x + 1)^2 = x + 3$$
$$x^2 + x - 2 = 0$$
$$(x - 1)(x + 2) = 0$$

With $x = 1$ we have $y = 4$, and with $x = -2$ we have $y = 1$.
Thus there are two equilibria: $(1, 4)$ and $(-2, 1)$.

(b) The Jacobian is $Df(x, y) = \begin{bmatrix} 2x + 2 & -1 \\ 1 & -1 \end{bmatrix}$, which has trace $\tau = 2x + 1$ and determinant $\delta = -2x - 1$.
With $(x, y) = (1, 4)$, we have $\tau = 3$ and $\delta = -3$, thus $(1, 4)$ is a *saddle*.
With $(x, y) = (-2, 1)$, we have $\tau = -3$ and $\delta = 3$, thus $(-2, 1)$ is a *stable focus* (notice $\frac{\tau^2}{4} < \delta$).

4. Suppose that $\mathbf{x}^*$ is an equilibrium of a four-dimensional system $\dot{\mathbf{x}} = f(\mathbf{x})$ with

$$Df(\mathbf{x}^*) = \begin{bmatrix} -1 & 1 & 1 & -2 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 4 & 3 \end{bmatrix}.$$

What are the dimensions of $W^s(\mathbf{x}^*)$ and $W^u(\mathbf{x}^*)$? Provide at least one sentence of explanation.

[4 marks]

The eigenvalues of $Df(\mathbf{x}^*)$ are -1, 2, and $3 \pm 4i$. Notice that $\mathbf{x}^*$ is hyperbolic. One eigenvalue has negative real part, thus $W^s(\mathbf{x}^*)$ is one-dimensional. Three eigenvalues have positive real part, thus $W^u(\mathbf{x}^*)$ is three-dimensional.

5. Consider the system

$$\begin{aligned} \dot{x} &= y + y^2 \ , \\ \dot{y} &= 2x - y + 3y^2 \ . \end{aligned}$$

For what values of a and b is the stable manifold of the origin $W^s(0,0)$ given by

$$y = ax + bx^2 + \mathcal{O}\big(x^3\big) \ ?$$

[10 marks]

At the origin, the Jacobian of f is

$$A = Df(0,0) = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}.$$

The characteristic polynomial is $\det(\lambda I - A) = \lambda^2 + \lambda - 2 = (\lambda - 1)(\lambda + 2)$. Thus $\lambda_1 = 1$ and $\lambda_2 = -2$ are eigenvalues. Notice the origin is hyperbolic and $W^s(0,0)$ is one-dimensional.

We see that the eigenvector corresponding to $\lambda_2 = -2$ is $v_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.
Thus $W^s(0,0)$ can be written as $y = \phi(x) = ax + bx^2 + \mathcal{O}(x^3)$ with $a = -2$.
Since $W^s(0,0)$ is an invariant, we have $\dot{y} = \phi'(x)\dot{x}$.
We have

$$\begin{aligned} \dot{y} &= 2x - \phi(x) + 3y^2 \\ &= 2x + 2x - bx^2 + 12x^2 + \mathcal{O}\big(x^3\big) \\ &= 4x + (12 - b)x^2 + \mathcal{O}\big(x^3\big) \end{aligned}$$

Also

$$\begin{aligned} \phi'(x)\dot{x} &= \big(-2 + 2bx + \mathcal{O}\big(x^2\big)\big)\left(-2x + bx^2 + 4x^2 + \mathcal{O}\big(x^3\big)\right) \\ &= 4x + (-6b - 8)\,x^2 + \mathcal{O}\big(x^3\big) \end{aligned}$$

Matching the x^2 coefficients gives

$$\begin{aligned}12 - b &= -6b - 8\\ 5b &= -20\\ b &= -4\end{aligned}$$

In summary,

$$a = -2\ , \qquad b = -4\ .$$

6. Determine the value of μ at which

$$\dot{x} = \mu + \tfrac{1}{2}x - x^4\ ,$$

has a saddle-node bifurcation.

[6 marks]

Equilibria are given implicitly by $\mu = x^4 - \frac{1}{2}x$.
Saddle-node bifurcations are where $\frac{d\mu}{dx} = 0$ (and $\frac{d^2\mu}{dx^2} \neq 0$).
We have $\frac{d\mu}{dx} = 4x^3 - \frac{1}{2}$ (and $\frac{d^2\mu}{dx^2} = 12x^2$).
Solving $\frac{d\mu}{dx} = 0$ gives $x^3 = \frac{1}{8}$ and so $x = \frac{1}{2}$ (and notice $\frac{d^2\mu}{dx^2} \neq 0$).
Here $\mu = \frac{1}{16} - \frac{1}{4} = \frac{-3}{16}$.

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