Surname: _____

First name: _____

Student ID: _____

MASSEY UNIVERSITY
Institute of Fundamental Sciences

160.734 (Studies in Applied Differential Equations)

Mid-Semester Test

Semester Two, 2018

Time allowed: **55 minutes**

Total marks: 40

This test is **closed book**. Calculators are permitted.

Attempt all questions. There are 5 questions altogether.

Show all working to receive full credit.
A blank page is provided at the back of the test, in case you need extra space.

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1. _____/10

2. _____/6

3. _____/11

4. _____/4

5. _____/9

Total:_____/40

1. Let $A = \begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix}$.

   (a) Compute e^{tA}.

   (b) Suppose $\mathbf{x}(t)$ is the solution to $\dot{\mathbf{x}} = A\mathbf{x}$ for which $\mathbf{x}(1) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.
   Use your answer to (a) to determine $\mathbf{x}(2)$.

[5 + 5 = 10 marks]

2. Let $\mathbf{x}^*$ be a equilibrium of $\dot{\mathbf{x}} = f(\mathbf{x})$, where $f : \mathbb{R}^n \to \mathbb{R}^n$ is C^1.

   (a) Define what it means for $\mathbf{x}^*$ to be Lyapunov stable.

   (b) Define what it means for $\mathbf{x}^*$ to be asymptotically stable.

[3 + 3 = 6 marks]

3. Let $\mathbf{x}^*(\mu)$ be an equilibrium of $\dot{\mathbf{x}} = f(\mathbf{x}; \mu)$, where $f : \mathbb{R}^4 \times \mathbb{R} \to \mathbb{R}^4$ is C^3.
   Suppose $W^s(\mathbf{x}^*(\mu))$ is three-dimensional for $\mu < 0$, and that $\mathbf{x}^*(\mu)$ undergoes a Hopf bifurcation at $\mu = 0$.
   What must be the dimension of $W^s(\mathbf{x}^*(\mu))$ for small $\mu > 0$? Why?

[4 marks]

4. Consider the system

$$\dot{x} = x^4 - 4x^3 + \mu.$$

(a) Determine the two values of μ for which the system has an equilibrium with a zero eigenvalue.

(b) Show that one of these is a saddle-node bifurcation and the other is not.

[4 + 5 = 9 marks]

5. Consider the system

$$\dot{x} = x - 2y,$$
$$\dot{y} = x^2 - 5y^2,$$

for which $(x, y) = (0, 0)$ is a non-hyperbolic equilibrium.

(a) The centre manifold $W^c(0, 0)$ can be written as $y = \alpha x + \beta x^2 + \mathcal{O}(x^3)$. Determine the values of α and β.

(b) On $W^c(0,0)$ we have $\dot{x} = \gamma x^2 + \mathcal{O}(x^3)$.
Determine the value of γ.

(c) Is $(x, y) = (0, 0)$ asymptotically stable? Why or why not?

[7 + 2 + 2 = 11 marks]

Use this page if you need extra space to answer the questions. If you use it, make a note on the page of the question that you have done so, and clearly indicate here which question you are answering.