Surname: ______________________

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MASSEY UNIVERSITY
Institute of Fundamental Sciences

160.734 (Studies in Applied Differential Equations)

Mid-Semester Test Solutions

Semester Two, 2018

Time allowed: **55 minutes**

Total marks: 40

This test is **closed book**. Calculators are permitted.

Attempt all questions. There are 5 questions altogether.

Show all working to receive full credit.
A blank page is provided at the back of the test, in case you need extra space.

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1. ______________/10

2. ______________/6

3. ______________/11

4. ______________/4

5. ______________/9

Total:____________/40

1. Let $A = \begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix}$.

   (a) Compute e^{tA}.

   We have $A = 2I + N$ where $N = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}$. Notice N^2 is the zero matrix, so N is nilpotent and $e^{tN} = I + tN$. Notice $2I$ and N commute, thus

$$\begin{aligned} e^{tA} &= e^{2tI} e^{tN} \\ &= e^{2t}(I + tN) \\ &= e^{2t} \begin{bmatrix} 1 & 0 \\ 3t & 1 \end{bmatrix}. \end{aligned}$$

   (b) Suppose $\mathbf{x}(t)$ is the solution to $\dot{\mathbf{x}} = A\mathbf{x}$ for which $\mathbf{x}(1) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.
   Use your answer to (a) to determine $\mathbf{x}(2)$.

   We have $\mathbf{x}(t) = e^{tA}\mathbf{x}(0)$, for all $t \in \mathbb{R}$.
   So $\mathbf{x}(1) = e^{A}\mathbf{x}(0)$ and $\mathbf{x}(2) = e^{2A}\mathbf{x}(0)$, hence

$$\begin{aligned} \mathbf{x}(2) &= e^{2A} e^{-A} \mathbf{x}(1) \\ &= e^{A}\mathbf{x}(1) \\ &= e^{2} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} 3\,e^2 \\ 13\,e^2 \end{bmatrix}. \end{aligned}$$

[5 + 5 = 10 marks]

2. Let $\mathbf{x}^*$ be a equilibrium of $\dot{\mathbf{x}} = f(\mathbf{x})$, where $f : \mathbb{R}^n \to \mathbb{R}^n$ is C^1.

   (a) Define what it means for $\mathbf{x}^*$ to be Lyapunov stable.

   That for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $\mathbf{x} \in B_\delta(\mathbf{x}^*)$ we have $\varphi_t(\mathbf{x}) \in B_\varepsilon(\mathbf{x}^*)$ for all $t \geq 0$.

   (b) Define what it means for $\mathbf{x}^*$ to be asymptotically stable.

   That $\mathbf{x}^*$ is Lyapunov stable and there exists $\eta > 0$ such that for all $\mathbf{x} \in B_\eta(\mathbf{x}^*)$ we have $\varphi_t(\mathbf{x}) \to \mathbf{x}^*$ as $t \to \infty$.

[3 + 3 = 6 marks]

3. Let $\mathbf{x}^*(\mu)$ be an equilibrium of $\dot{\mathbf{x}} = f(\mathbf{x}; \mu)$, where $f : \mathbb{R}^4 \times \mathbb{R} \to \mathbb{R}^4$ is C^3.
   Suppose $W^s(\mathbf{x}^*(\mu))$ is three-dimensional for $\mu < 0$, and that $\mathbf{x}^*(\mu)$ undergoes a Hopf bifurcation at $\mu = 0$.
   What must be the dimension of $W^s(\mathbf{x}^*(\mu))$ for small $\mu > 0$? Why?

   For a Hopf bifurcation, the dimension changes by two.
   The dimension cannot change from 3 to 5 because the system is only four-dimensional.
   Thus the dimension must change from 3 to 1.

[4 marks]

4. Consider the system
$$\dot{x} = x^4 - 4x^3 + \mu.$$

(a) Determine the two values of μ for which the system has an equilibrium with a zero eigenvalue.

We have
$$\frac{\partial f}{\partial x} = 4x^3 - 12x^2 = 4x^2(x-3),$$
which equals 0 when $x = 0$ or $x = 3$.
Firstly solving $f(x;\mu) = 0$ using $x = 0$ gives $\mu = 0$.
Secondly solving $f(x;\mu) = 0$ using $x = 3$ gives $\mu = 27$.

(b) Show that one of these is a saddle-node bifurcation and the other is not.

We have
$$\frac{\partial^2 f}{\partial x^2} = 12x^2 - 24x.$$
Firstly observe $\frac{\partial^2 f}{\partial x^2}\Big|_{x=0} = 0$, thus $\mu = 0$ is not a saddle-node bifurcation.
Secondly observe $\frac{\partial^2 f}{\partial x^2}\Big|_{x=3} = 36 \neq 0$, and $\frac{\partial f}{\partial \mu}\Big|_{x=3} = 1 \neq 0$, thus $\mu = 0$ is a saddle-node bifurcation.

[4 + 5 = 9 marks]

5. Consider the system

$$\dot{x} = x - 2y,$$
$$\dot{y} = x^2 - 5y^2,$$

for which $(x, y) = (0, 0)$ is a non-hyperbolic equilibrium.

(a) The centre manifold $W^c(0,0)$ can be written as $y = \alpha x + \beta x^2 + \mathcal{O}(x^3)$. Determine the values of α and β.

We have $Df(0,0) = \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$.

The eigenvalue 0 has corresponding eigenvector $v = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$, thus $\alpha = \frac{1}{2}$.

Thus on $W^c(0,0)$,

$$\begin{aligned} \dot{y} &= x^2 - 5\left(\frac{1}{4}x^2 + \mathcal{O}(x^3)\right) \\ &= -\frac{1}{4}x^2 + \mathcal{O}(x^3). \end{aligned}$$

Also, since $W^c(0,0)$ is invariant,

$$\begin{aligned} \dot{y} &= \frac{dy}{dx}\,\dot{x} \\ &= \left(\frac{1}{2} + 2\beta x + \mathcal{O}(x^2)\right)\left(x - 2\left(\frac{1}{2}x + \beta x^2 + \mathcal{O}(x^3)\right)\right) \\ &= -\beta x^2 + \mathcal{O}(x^3). \end{aligned}$$

By matching these we obtain $\beta = \frac{1}{4}$.

(b) On $W^c(0,0)$ we have $\dot{x} = \gamma x^2 + \mathcal{O}(x^3)$.
Determine the value of γ.

We have

$$\begin{aligned}\dot{x} &= x - 2\left(\frac{1}{2}x + \frac{1}{4}x^2 + \mathcal{O}\big(x^3\big)\right) \\ &= -\frac{1}{2}x^2 + \mathcal{O}\big(x^3\big).\end{aligned}$$

(c) Is $(x,y) = (0,0)$ asymptotically stable? Why or why not?

No, regardless of the dynamics on $W^c(0,0)$ the other eigenvalue of $Df(0,0)$ is $\lambda = 1$ which has positive real part so $(0,0)$ is unstable.

[$\mathbf{7 + 2 + 2 = 11}$ marks]

Use this page if you need extra space to answer the questions. If you use it, make a note on the page of the question that you have done so, and clearly indicate here which question you are answering.