Course Outline for 160.734: Studies in Applied Differential Equations

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Instructor:

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Assessment:

I)	Assignment 1	15%
	Test	35%
II)	Assignment 2	13%
	Assignment 3	13%
	Assignment 4	24%

- Each assignment will require the use of MATLAB (or alternate but similar software if you prefer).

Important dates:

Assignment 1 due:	Monday, August 5
Assignment 2 due:	Monday, August 26
Assignment 3 due:	Monday, September 23
Test:	Monday, September 30
Assignment 4 due:	Tuesday, October 29

Topics:
This is a course on *dynamical systems*. Mostly we will study ODEs: $\dot{\mathbf{x}} = f(\mathbf{x})$ (parts 1–6); toward the end we will study maps: $\mathbf{x}_{i+1} = f(\mathbf{x}_i)$ (parts 7–8).

1) Linear systems of ODEs; matrix exponentiation; fundamental solution theorem; dealing with complex eigenvalues and repeated eigenvalues; invariant subspaces
2) Existence and uniqueness of solutions to ODEs; Grönwall's inequality; dependence on initial conditions and parameters
3) Linearisation; hyperbolicity; stability; Lyapunov functions; topological equivalence; Hartman-Grobman theorem; ω-limit sets; Poincaré-Bendixson theorem; attractors
4) Stable, unstable and centre manifolds and associated theorems
5) Structural stability; saddle-node and Hopf bifurcations; extended centre manifolds; global bifurcations
6) Chaos; fractals; Poincaré maps
7) Linear maps; one-dimensional maps; saddle-node, period-doubling and Neimark-Sacker bifurcations; logistic map
8) Symbolic dynamics; kneading theory; introduction to measure theory and ergodic theory

Reference material:
There is no set textbook; notes will be provided. Massey library has several books that you could use to aid your studies. Foremost is the excellent book of Meiss [1] (*515.39 Mei*) which covers ODEs and is pitched at our level of detail and difficulty. Indeed several sections of the notes follow this book quite closely. Currently the library has one print copy at the Albany campus (you can request it!).
Suitable books available on the Manawatu campus (as of February 2017) are listed below. The parts of the course that each book is particularly useful for are listed in angled brackets.

- Alligood, Sauer, Yorke [2] (*003.85 All*) $\langle 4,5,6,7\rangle$
- Arrowsmith, Place [3] (*515.352 Arr*) $\langle 1,3\rangle$
- Chicone [4] (*515.35 Chi*) $\langle 2,3,4,5\rangle$
- Devaney [5] (*515.352 Dev*) $\langle 7\rangle$
- Elaydi [6] (*515.39 Ela*) $\langle 7\rangle$
- Glendinning [7] (*515.355 Gle*) $\langle 3,4,5,7\rangle$
- Hale, Koçak [8] (*515.35 Hal*) $\langle 3,4,5,7\rangle$
- Hirsch, Smale, Devaney [9] (*515.35 Hir*) $\langle 1,2,3,6,7\rangle$
- Katok, Hasselblatt [10] (*515.352 Kat*) $\langle 4,7,8\rangle$
- Lasota, Mackey [11] (*003.75 Las*) $\langle 8\rangle$
- Martelli [12] (*003.85 Mar*) $\langle 7\rangle$
- Perko [13] (*515.355 Per*) $\langle 1,2,3,4,5\rangle$
- Robinson [14] (*514.74 Rob*) $\langle 6,7\rangle$
- Robinson [15] (*515.39 Rob*) $\langle 6,7\rangle$

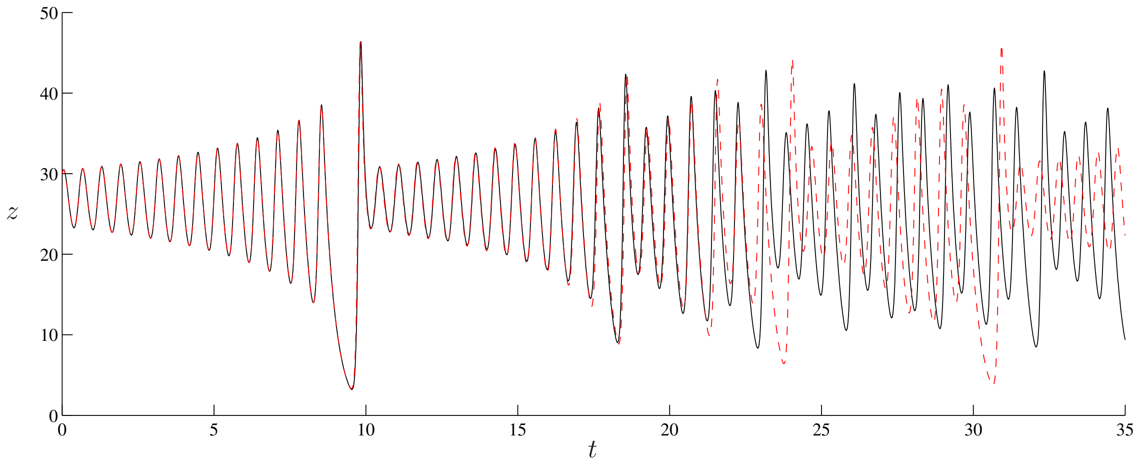

Figure 1: Time series of (1) for two slightly different initial conditions.

- Sternberg [16] (*515.39 Ste*) ⟨7, 8⟩

Some motivation:
In 1963 Ed Lorenz working at MIT introduced the following three-dimensional system of ODEs to model convection in the atmosphere [17]:

$$\begin{aligned} \dot{x} &= \sigma(y - x), \\ \dot{y} &= rx - y - xz, \\ \dot{z} &= xy - bz. \end{aligned} \tag{1}$$

The quantities x, y, and z are the *variables* of the system — these are functions of time, t. They represent, in a simplified way, the strength and directions of the convection. The quantities σ, r, and b are the *parameters* of the system — these are constant in t. They represent physical properties of the atmosphere.
The black curve in Fig. 1 is a *time series* of the solution to (1) using the initial condition $(x(0), y(0), z(0)) = (10, 10, 30)$ and the parameter values

$$\sigma = 10, \qquad r = 28, \qquad b = \frac{8}{3}. \tag{2}$$

Fig. 2 shows a *phase portrait* of the same solution. These were computed numerically using a finite difference scheme (ODE45 in MATLAB).
We observe that the solution settles to oscillations of rather irregular amplitude. This is the *attractor* of the system and for the given parameter values is unique. That is, for almost any initial condition the solution settles to these irregular oscillations.

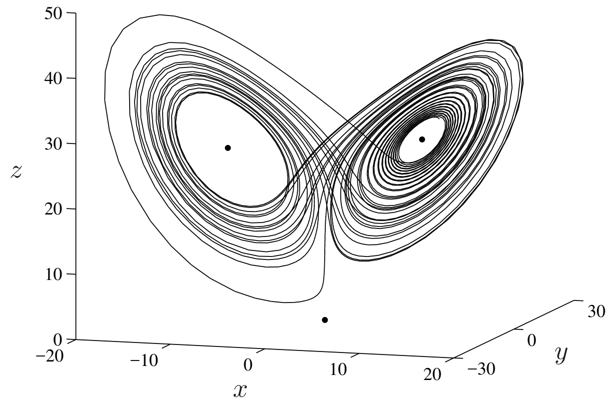

Figure 2: A phase portrait of (1). The three dots are equilibria of (1).

The dashed curve in Fig. 1 is a time series for the slightly different initial condition $(x(0), y(0), z(0)) = (10, 10, 30.0001)$. Remarkably, the difference in the two time series eventually becomes large. This is not due to numerical error. Nor does it occur before the two solutions become close to the attractor. The large difference occurs because on the attractor nearby solutions are locally repelling in spite of the fact that the attractor is globally attracting.
This is the essence of chaos. For the purposes of predicting future events, this is not just problematic: it is a fundamental barrier to the limits of our predictive power. From a practical viewpoint, our knowledge of the initial condition must involve

some error. Regardless of the accuracy of our mathematical model, chaos causes small measurement errors to quickly grow into hopelessly large errors. From [17]: *In view of the inevitable inaccuracy and incompleteness of weather observations, precise very-long range forecasting would seem to be non-existent.*

But all is not lost. It can be useful to study the *statistical* properties of a chaotic attractor, and indeed chaos has many practical applications, such as to robust and secure signal transmission.

For the theory that we will develop, the primary goal is to be able to characterise "the dynamics" of a given system. Often short-term transient dynamics is not important to us, in which case it suffices to determine the attractors of the system, describe their basins of attraction, and understand how the attractors change under parameter variation.

References

[1] J.D. Meiss. *Differential Dynamical Systems.* SIAM, Philadelphia, 2007.

[2] K.T. Alligood, T.D. Sauer, and J.A. Yorke. *Chaos. An Introduction to Dynamical Systems.* Springer, New York, 1997.

[3] D.K. Arrowsmith and C.M. Place. *Dynamical Systems. Differential equations, maps and chaotic behaviour.* Chapman and Hall., Boca Raton, FL, 1992.

[4] C. Chicone. *Ordinary Differential Equations with Applications.* Springer, New York, 1999.

[5] R.L. Devaney. *A First Course in Chaotic Dynamical Systems. Theory and Experiment.* Addison-Wesley, New York, 1992.

[6] S.N. Elaydi. *Discrete Chaos with Applications in Science and Engineering.* Chapman and Hall., Boca Raton, FL, 2008.

[7] P. Glendinning. *Stability, Instability and Chaos: An Introduction to the Theory of Nonlinear Differential Equations.* Cambridge University Press, New York, 1999.

[8] J. Hale and H. Koçak. *Dynamics and Bifurcations.* Springer-Verlag, New York, 1991.

[9] M.W. Hirsch, S. Smale, and R.L. Devaney. *Differential Equations, Dynamical Systems, and an Introduction to Chaos.* Elsevier, New York, 2013.

[10] A. Katok and B. Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems.* Cambridge University Press, New York, 1995.

[11] A. Lasota and M.C. Mackey. *Chaos, Fractals, and Noise. Stochastic Aspects of Dynamics.* Springer-Verlag, New York, 1994.

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