Notes for 160.734
Part I: Linear Systems

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Our journey towards a qualitative and quantitative understanding of systems of nonlinear dynamical systems begins with linear ODEs. This is in part because many methods that we will encounter later in the course involve approximating nonlinear ODEs with linear ones.

Unlike nonlinear ODEs, the solution to a system of linear ODEs can always be written down explicitly. We will first see that eigenvalues and eigenvectors are central to achieving this, and sometimes we need so-called generalised eigenvectors to obtain the general solution, but our main tool will be that of matrix exponentiation.

These notes closely follow chapter 2 of Meiss [1].

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1 Introduction

- A system of linear ODEs can be written as

$$\dot{\mathbf{x}} = A\mathbf{x}, \tag{1.1}$$

  where $\mathbf{x} = \mathbf{x}(t) \in \mathbb{R}^n$, A is a real-valued $n \times n$ matrix, and the dot denotes differentiation with respect to time, t. Often we're interested in the solution to (1.1) subject to $\mathbf{x}(0) = \mathbf{x}_0$, where $\mathbf{x}_0$ is a given initial condition.

- Suppose $\mathbf{x}_0 = v$, where v is an eigenvector of A corresponding to an eigenvalue $\lambda \in \mathbb{R}$. In this case the vector field at $\mathbf{x}_0$, namely $A\mathbf{x}_0$, points in the same direction as v. Thus as the solution $\mathbf{x}(t)$ evolves it continues to have direction v. We can therefore write $\mathbf{x}(t) = c(t)v$, where $c : \mathbb{R} \to \mathbb{R}$ with $c(0) = 1$. Substituting this into (1.1) leads to

$$\dot{c}(t)v = \lambda c(t)v,$$

  and so

$$\dot{c}(t) = \lambda c(t). \tag{1.2}$$

  The general solution to the one-dimensional ODE (1.2) is

$$c(t) = k\mathrm{e}^{\lambda t}, \tag{1.3}$$

  where $k \in \mathbb{R}$. Thus the solution to (1.1) with $\mathbf{x}(0) = v$ is

$$\mathbf{x}(t) = \mathrm{e}^{\lambda t}v. \tag{1.4}$$

- Since (1.1) is *linear*, if $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are solutions to (1.1), then for any constants c_1 and c_2, the linear combination

$$c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t), \tag{1.5}$$

  is also a solution to (1.1).

- If we can find m linearly independent eigenvectors, we can form an m-dimensional family of solutions. Ideally we'd like an n-dimensional family of solutions so that we can construct the particular solution for any initial condition $\mathbf{x}_0$.

Definition 1.1. An *equilibrium* of a nonlinear system of ODEs, $\dot{\mathbf{x}} = f(\mathbf{x})$, is a point $\mathbf{x}^*$ for which $f(\mathbf{x}^*) = \mathbf{0}$.

- If $\mathbf{x}(0) = \mathbf{x}^*$, then $\mathbf{x}(t) = \mathbf{x}^*$ is a solution for all t.
- Note that $\mathbf{0}$ (the zero vector or origin) is always an equilibrium of the linear system (1.1).

2 Two-dimensional linear systems

- Here we study (1.1) in the case $n = 2$ and write

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

  The characteristic polynomial of A is

$$\det(\lambda I - A) = \lambda^2 - \tau\lambda + \delta, \tag{2.1}$$

  where

$$\tau = a + d, \qquad \delta = ad - bc, \tag{2.2}$$

are the trace and determinant of A. The eigenvalues of A are

$$\lambda_{\pm} = \frac{\tau \pm \sqrt{\tau^2 - 4\delta}}{2}. \tag{2.3}$$

- As long as $\lambda_+ \neq \lambda_-$, we can be sure that λ_+ and λ_- have linearly independent eigenvectors v_+ and v_- (possibly complex-valued). In this case the linear combination

$$\mathbf{x}(t) = c_+ \mathrm{e}^{\lambda_+ t} v_+ + c_- \mathrm{e}^{\lambda_- t} v_- \,, \tag{2.4}$$

where $c_+, c_- \in \mathbb{C}$, represents the most general solution to (1.1).

Exercise 2.1. If λ_+ and λ_- are complex-valued, then v_+ and v_- are also complex-valued. Since A is real-valued, $\lambda_- = \overline{\lambda}_+$ and $v_- = \overline{v}_+$ (where bars denote complex conjugation). Show that in this case if we set $c_- = \overline{c}_+$, then (2.4) will be real-valued (which is what we want).

- Since the eigenvalues are completely determined by τ and δ, we can classify the basic properties of (1.1) in two dimensions just from the values of τ and δ. However, with an eye to our eventual study of nonlinear systems, we say that we are classifying, not the system (1.1), but the equilibrium $\mathbf{x} = \mathbf{0}$.

i) If $\tau > 0$ and $0 < \delta < \frac{\tau^2}{4}$, then $0 < \lambda_- < \lambda_+$. Here both terms in (2.4) blow up as $t \to \infty$ and we refer to $\mathbf{0}$ as an *unstable node.*

ii) If $\tau < 0$ and $0 < \delta < \frac{\tau^2}{4}$, then $\lambda_- < \lambda_+ < 0$. Here both terms in (2.4) tend to zero as $t \to \infty$ and we refer to $\mathbf{0}$ as a *stable node.*

iii) If $\delta < 0$, then $\lambda_- < 0 < \lambda_+$. Here $\mathrm{e}^{\lambda_+ t} \to \infty$ whereas $\mathrm{e}^{\lambda_- t} \to 0$, as $t \to \infty$ and we refer to $\mathbf{0}$ as a *saddle.*

iv) If $\tau > 0$ and $\delta > \frac{\tau^2}{4}$, then λ_+ and λ_- are complex-valued with positive real parts. Here solutions (2.4) spiral outwards as t increases and we refer to $\mathbf{0}$ as an *unstable focus.*

v) If $\tau < 0$ and $\delta > \frac{\tau^2}{4}$, then λ_+ and λ_- are complex-valued with negative real parts. Here solutions (2.4) spiral inwards as t increases and we refer to $\mathbf{0}$ as a *stable focus.*

- If $\tau = 0$ and $\delta > 0$, then $\lambda_{\pm} = \pm \mathrm{i}\sqrt{\delta}$ are purely imaginary. Here solutions (2.4) form ellipses and we refer to $\mathbf{0}$ as a *centre.*
- In other cases if we need to refer to $\mathbf{0}$ as something we can just say it is *degenerate.*
- If $\delta = \frac{\tau^2}{4}$, then A has a repeated eigenvalue. The basic structure of the solutions depends on whether or not A has one or two linearly independent eigenvectors (usually there is just one).
- If $\delta = 0$, then A has a zero eigenvalue.

Exercise 2.2. Show that if A has a zero eigenvalue then the corresponding eigenvector generates a line of equilibria.

3 Matrix exponentiation

Here we show how to take e to the power of a matrix. In the next section we use this to solve the linear system (1.1) in a concise manner.

Definition 3.1. Let A be a real-valued $n \times n$ matrix. Then

$$\mathrm{e}^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}. \tag{3.1}$$

- The following result follows easily from the above definition.

Lemma 3.1. *Let*

$$D = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix}$$

be a diagonal matrix. Then

$$\mathrm{e}^D = \begin{bmatrix} \mathrm{e}^{d_1} & & \\ & \ddots & \\ & & \mathrm{e}^{d_n} \end{bmatrix}. \tag{3.2}$$

- Given numbers x and y, we would happily rewrite the product $\mathrm{e}^x \mathrm{e}^y$ as e^{x+y}. However, for matrices we cannot always do this. The next result tells us that we can do this if the matrices commute (for an outline of a proof of this result see [1], chapter 2, exercise 6).

Lemma 3.2. *If $AB = BA$, then $\mathrm{e}^A \mathrm{e}^B = \mathrm{e}^{A+B}$.*

Exercise 3.1. Let $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Show that for any $t \in \mathbb{R}$,

$$\mathrm{e}^{tA} = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}.$$

Exercise 3.2. Let $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$, where $a, b \in \mathbb{R}$. Show that for any $t \in \mathbb{R}$,

$$\mathrm{e}^{tA} = \mathrm{e}^{at} \begin{bmatrix} \cos(bt) & \sin(bt) \\ -\sin(bt) & \cos(bt) \end{bmatrix}. \tag{3.3}$$

HINT: write $A = aI + b\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and use the result of the previous exercise and Lemma 3.2.

Definition 3.2. A real-valued $n \times n$ matrix N is said to be *nilpotent* if $N^k = 0$ (the zero matrix) for some $k \geq 0$.

Exercise 3.3. Show that 0 is the only eigenvalue of a nilpotent matrix.

- For a nilpotent matrix N, it is straight-forward to compute e^N because the series (3.1) terminates.

4 Fundamental solution theorem

Theorem 4.1. *The unique solution to* (1.1) *with* $\mathbf{x}(0) = \mathbf{x}_0$ *is*

$$\mathbf{x}(t) = \mathrm{e}^{tA}\mathbf{x}_0 \,. \tag{4.1}$$

Proof. First we show that (4.1) is a solution. From the definition (3.1) one can show that $\frac{d}{dt}\mathrm{e}^{tA} = A\mathrm{e}^{tA}$ (as we would expect), and so

$$\dot{\mathbf{x}}(t) = \frac{d}{dt}\mathrm{e}^{tA}\mathbf{x}_0 = A\mathrm{e}^{tA}\mathbf{x}_0 = A\mathbf{x}(t).$$

Also $\mathbf{x}(0) = \mathrm{e}^0\mathbf{x}_0 = \mathbf{x}_0$, hence (4.1) is indeed a solution.

Second we verify uniqueness. Let $\mathbf{y}(t)$ be another solution. Then by the product rule

$$\frac{d}{dt}\left(\mathrm{e}^{-tA}\mathbf{y}(t)\right) = -\mathrm{e}^{-tA}A\mathbf{y}(t) + \mathrm{e}^{-tA}\dot{y}(t).$$

The right hand-side of this equation is zero because $\mathbf{y}(t)$ is a solution to (1.1) by assumption. Therefore $\mathrm{e}^{-tA}\mathbf{y}(t)$ equals a constant, call it $\mathbf{y}_0$. That is, $\mathbf{y}(t) = \mathrm{e}^{tA}\mathbf{y}_0$. But $\mathbf{y}(0) = \mathbf{x}_0$, since $\mathbf{y}(t)$ satisfies the initial condition by assumption. Thus $\mathbf{y}_0 = \mathbf{x}_0$ and hence $\mathbf{y}(t) = \mathbf{x}(t)$. That is, $\mathbf{x}(t)$ is unique. □

- If A is diagonalisable, then we can compute e^{tA} by substituting $A = P\Lambda P^{-1}$, where Λ is diagonal, into (3.1) to obtain

$$\begin{aligned} \mathrm{e}^{tA} &= \sum_{k=0}^{\infty} \frac{t^k}{k!} \underbrace{\left(P\Lambda P^{-1}\right)\left(P\Lambda P^{-1}\right)\cdots\left(P\Lambda P^{-1}\right)}_{k \text{ times}} \\ &= P\sum_{k=0}^{\infty}\frac{t^k\Lambda^k}{k!}P^{-1} \\ &= P\mathrm{e}^{t\Lambda}P^{-1}, \end{aligned}$$

and $\mathrm{e}^{t\Lambda}$ can be evaluated by (3.2).

- Consider the *affine* system

$$\dot{\mathbf{x}} = A\mathbf{x} + b, \tag{4.2}$$

where $b \in \mathbb{R}^n$. If A is invertible (4.2) has the unique equilibrium $\mathbf{x}^* = -A^{-1}b$ and the substitution $\mathbf{y} = \mathbf{x} - \mathbf{x}^*$ produces $\dot{\mathbf{y}} = A\mathbf{y}$. Consequently the solution to (4.2) is

$$\mathbf{x}(t) = \mathrm{e}^{tA}(\mathbf{x}_0 - \mathbf{x}^*) + \mathbf{x}^*. \tag{4.3}$$

If A is not invertible we can instead use the formula

$$\mathbf{x}(t) = \mathrm{e}^{tA}\mathbf{x}_0 + \int_0^t \mathrm{e}^{sA}b\,ds. \tag{4.4}$$

Exercise 4.1. Show that (4.3) and (4.4) are the same when A is invertible.

5 Generalised eigenvectors and tips for computing e^{tA}

We begin by reviewing some basic concepts related to eigenvalues, then introduce generalised eigenvectors.

Definition 5.1. Let A be an $n \times n$ matrix.

- An *eigenvalue* of A is a number $\lambda \in \mathbb{C}$ for which $\det(\lambda I - A) = 0$.
- The *algebraic multiplicity* of λ is the unique number k for which $\det(tI - A) = (t - \lambda)^k q(t)$, where $q(\lambda) \neq 0$.
- The *eigenspace* associated with λ is the nullspace of $\lambda I - A$. Any nonzero vector in the eigenspace is said to be an *eigenvector* associated with λ.

- The *geometric multiplicity* of λ is the dimension of the eigenspace.

- Let $\lambda_1, \ldots, \lambda_m$ be the eigenvalues of an $n \times n$ matrix A, let $k_1, \ldots, k_m$ be their algebraic multiplicities, and let $\ell_1, \ldots, \ell_m$ be their geometric multiplicities. Since $\det(tI - A)$ is a degree-n polynomial, by the fundamental theorem of algebra we have

$$k_1 + \cdots + k_m = n.$$

Also $1 \leq \ell_i \leq k_i$, for each i.

- If $\ell_1 + \cdots + \ell_m = n$, then A has n linearly independent eigenvectors. If P is a matrix whose columns are these eigenvectors, then

$$P^{-1}AP = \Lambda \,,$$

where Λ is the diagonal matrix where each (i,i)-element of Λ is the eigenvalue corresponding to the i^{th} column of P.

- So if $\ell_1 + \cdots + \ell_m = n$ we can compute e^{tA} by using P to diagonalise A (see the first example in §6).

- If $\ell_1 + \cdots + \ell_m < n$ we can instead compute e^{tA} by using so-called generalised eigenvectors.

Definition 5.2. Let $\lambda \in \mathbb{C}$ be an eigenvalue of an $n \times n$ matrix A and let k be its algebraic multiplicity. The *generalised eigenspace* associated with λ is the nullspace of $(\lambda I - A)^k$. Any nonzero vector in the generalised eigenspace is said to be a *generalised eigenvector* associated with λ.

Lemma 5.1. *Let $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$ be the eigenvalues of an $n \times n$ matrix, and let $E_1, \ldots, E_m$ be the corresponding generalised eigenspaces. Then the dimension of each E_i is equal to the algebraic multiplicity of λ_i, and*[1]

$$E_1 \oplus \cdots \oplus E_m = \mathbb{C}^n. \tag{5.1}$$

- Lemma 5.1 is proved in many standard algebra textbooks; refer to the references listed on pg. 50 of [1].

- Lemma 5.1 tells that the generalised eigenspaces provide us with enough vectors to form a nonsingular matrix $\tilde{P}$. Then to compute e^{tA} we work with $\tilde{\Lambda} = \tilde{P}^{-1}A\tilde{P}$ (see the third example in §6).

- A second key property of generalised eigenspaces is that they are 'invariant' as indicated in the next exercise (complex eigenvalues are dealt with in Exercise 5.2). Invariant structures are crucial to our understanding of nonlinear systems of ODEs as we will see later in the course.

Exercise 5.1. Let $\lambda \in \mathbb{R}$ be an eigenvalue of a real-valued matrix A, and let $E \subset \mathbb{R}^n$ be its corresponding generalised eigenspace. Let $\mathbf{x}(t)$ be the solution to (1.1) given $\mathbf{x}(0) = \mathbf{x}_0 \in E$. Show that $\mathbf{x}(t) \in E$ for all $t \in \mathbb{R}$. Hint: first show that $v \in E$ implies $Av \in E$.

- If some eigenvalues of A are complex-valued, it is instead simpler to work with a matrix $\tilde{P}$ that replaces complex conjugate pairs of generalised eigenvectors with their real and imaginary parts (see the second example in §6).

Exercise 5.2. Let $\lambda \notin \mathbb{R}$ be an eigenvalue of a real-valued matrix A, and let E be its corresponding generalised eigenspace. Let $\hat{E} = \mathrm{Re}(E) \oplus \mathrm{Im}(E) \subset \mathbb{R}^n$. Let $\mathbf{x}(t)$ be the solution to (1.1) given $\mathbf{x}(0) = \mathbf{x}_0 \in \hat{E}$. Show that $\mathbf{x}(t) \in \hat{E}$ for all $t \in \mathbb{R}$.

- An excess of alternate approaches to computing e^{tA} is given in [2].

6 Examples of computing e^{tA}

Example 6.1. Let $A = \begin{bmatrix} -3 & -2 \\ -2 & -3 \end{bmatrix}$.
The eigenvalues and eigenvectors are:

$$\lambda_1 = -1, \quad v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

$$\lambda_2 = -5, \quad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Let $P = \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$.

Then $P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and

[1]The symbol $\oplus$ denotes the *direct sum* which for subspaces is defined as follows. For any two subspaces U_1 and U_2, their sum is defined as $U_1 + U_2 = \{u_1 + u_2 \mid u_1 \in U_1, u_2 \in U_2\}$. Whenever we write $V = U_1 \oplus U_2$ we are saying that $V = U_1 + U_2$ and $U_1 \cap U_2 = \{\mathbf{0}\}$.

$$\Lambda = P^{-1}AP = \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -5 \end{bmatrix}.$$

Then

$$\begin{aligned} e^{tA} &= Pe^{t\Lambda}P^{-1} \\ &= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-5t} \end{bmatrix} \tfrac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \\ &= \tfrac{1}{2} \begin{bmatrix} e^{-t} + e^{-5t} & -e^{-t} + e^{-5t} \\ -e^{-t} + e^{-5t} & e^{-t} + e^{-5t} \end{bmatrix}. \end{aligned}$$

Example 6.2. Let $A = \begin{bmatrix} 0 & -2 \\ 1 & 2 \end{bmatrix}$.
The eigenvalues and eigenvectors are

$$\lambda = 1 + i, \qquad v = \begin{bmatrix} -1 + i \\ 1 \end{bmatrix}, \tag{6.1}$$

and the complex conjugates of these.
We can compute e^{tA} by using $P = \begin{bmatrix} v & \bar{v} \end{bmatrix}$ as in the previous example, but the algebra gets messy because the numbers are complex-valued. It is easier to let $\tilde{P} = \begin{bmatrix} \mathrm{Re}(v) & \mathrm{Im}(v) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$. Then $\tilde{P}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ and $\tilde{\Lambda} = \tilde{P}^{-1}A\tilde{P} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. By using (3.3) we obtain

$$\begin{aligned} e^{tA} &= \tilde{P}e^{t\tilde{\Lambda}}\tilde{P}^{-1} \\ &= \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} e^{t} \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \\ &= e^{t} \begin{bmatrix} \cos(t) - \sin(t) & -2\sin(t) \\ \sin(t) & \cos(t) + \sin(t) \end{bmatrix}. \end{aligned}$$

Example 6.3. Let $A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix}$. The eigenvalues are $\lambda_1 = -2$, with algebraic multiplicity 1 and eigenvector $v_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, and $\lambda_2 = 2$, with algebraic multiplicity 2. If we search for eigenvectors corresponding to λ_2 we will find that there is only one (more precisely the eigenspace is one-dimensional). Thus we instead compute a basis for the generalised eigenspace: the nullspace of

$$(\lambda_2 I - A)^2 = \begin{bmatrix} 8 & 0 & -8 \\ 0 & 0 & 0 \\ -8 & 0 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where $\sim$ means 'is row equivalent to'. We can see that the vectors $w_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $w_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ form such a basis. We then let $\tilde{P} = \begin{bmatrix} v_1 & w_1 & w_2 \end{bmatrix}$, which leads to

$$\tilde{\Lambda} = \tilde{P}^{-1}A\tilde{P} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

Next, write $\tilde{\Lambda} = D + B$ where

$$D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

This separation allows us to easily compute e^{tA} because D is diagonal, B is nilpotent ($B^2 = 0$, and so $e^{tB} = I + tB$), and D and B commute (check this!). Specifically we have

$$\begin{aligned} e^{tA} &= \tilde{P}e^{t\tilde{\Lambda}}\tilde{P}^{-1} \\ &= \tilde{P}e^{t(D+B)}\tilde{P}^{-1} \\ &= \tilde{P}e^{tD}e^{tB}\tilde{P}^{-1} \\ &\vdots \\ &= \begin{bmatrix} \cosh(2t) & te^{2t} & \sinh(2t) \\ 0 & e^{2t} & 0 \\ \sinh(2t) & te^{2t} & \cosh(2t) \end{bmatrix}. \end{aligned}$$

7 Stable, unstable, and centre subspaces

Definition 7.1. Consider the linear system (1.1) and let $v_1, \ldots, v_n$ be a set of linearly independent generalised eigenvectors of A. For each $j = 1, \ldots, n$, decompose v_j into its real and imaginary parts as

$$v_j = u_j + iw_j\ , \tag{7.1}$$

and let λ_j denote the corresponding eigenvalue (if $\lambda_j \in \mathbb{R}$ then $w_j = \mathbf{0}$).

i) The *stable subspace* of $\mathbf{0}$, denoted $E^s(\mathbf{0})$, is the span of all u_j and w_j for which $\mathrm{Re}(\lambda_j) < 0$.
ii) The *unstable subspace* of $\mathbf{0}$, denoted $E^u(\mathbf{0})$, is the span of all u_j and w_j for which $\mathrm{Re}(\lambda_j) > 0$.
iii) The *centre subspace* of $\mathbf{0}$, denoted $E^c(\mathbf{0})$, is the span of all u_j and w_j for which $\mathrm{Re}(\lambda_j) = 0$.

- For any $\mathbf{x}_0 \in E^s(\mathbf{0})$, we have $\mathrm{e}^{tA}\mathbf{x}_0 \in E^s(\mathbf{0})$ for all $t \in \mathbb{R}$ (this follows from Exercises 5.1 and 5.2). Also $\mathrm{e}^{tA}\mathbf{x}_0 \to \mathbf{0}$ as $t \to \infty$ because all associated eigenvalues have negative real part. Here we provide a uniform exponentially decaying bound on the norm of $\mathrm{e}^{tA}\mathbf{x}_0$.

  Let $\alpha > 0$ be such that $\mathrm{Re}(\lambda_j) < -\alpha$ for each λ_j with $\mathrm{Re}(\lambda_j) < 0$. It can be shown that there exists $K_s \in \mathbb{R}$ such that for every $\mathbf{x}_s \in E^s(\mathbf{0})$ we have
$$\left\|\mathrm{e}^{tA}\mathbf{x}_s\right\| \le K_s\,\mathrm{e}^{-\alpha t}\|\mathbf{x}_s\|, \tag{7.2}$$
  for all $t \ge 0$ (see [1], pages 58–59).

- Similarly if $\alpha > 0$ is such that $\mathrm{Re}(\lambda_j) > \alpha$ for each λ_j with $\mathrm{Re}(\lambda_j) > 0$, then there exists $K_u \in \mathbb{R}$ such that for every $\mathbf{x}_u \in E^u(\mathbf{0})$ we have
$$\left\|\mathrm{e}^{-tA}\mathbf{x}_u\right\| \le K_u\,\mathrm{e}^{-\alpha t}\|\mathbf{x}_u\|, \tag{7.3}$$
  for all $t \ge 0$.

- Since A has n linearly independent generalised eigenvectors, the span of $E^s(\mathbf{0})$, $E^u(\mathbf{0})$, and $E^c(\mathbf{0})$ is $\mathbb{R}^n$. Since the sum of the dimensions of $E^s(\mathbf{0})$, $E^u(\mathbf{0})$, and $E^c(\mathbf{0})$ is n, the intersection of any two of these subspaces is $\{\mathbf{0}\}$, and thus we can write
$$\mathbb{R}^n = E^s(\mathbf{0}) \oplus E^u(\mathbf{0}) \oplus E^c(\mathbf{0}). \tag{7.4}$$

- This tells us that every $\mathbf{x} \in \mathbb{R}^n$ can be uniquely written as $\mathbf{x} = \mathbf{x}_s + \mathbf{x}_u + \mathbf{x}_c$, where $\mathbf{x}_s \in E^s(\mathbf{0})$, $\mathbf{x}_u \in E^u(\mathbf{0})$, and $\mathbf{x}_c \in E^c(\mathbf{0})$.

Example 7.1. Compute the stable, unstable, and centre subspaces of $\mathbf{0}$ with
$$A = \begin{bmatrix} 1 & 1 & -4 & 2 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

The eigenvalues of A are $\lambda_1 = -1 + \mathrm{i}$, $\lambda_2 = \overline{\lambda}_1$, and $\lambda_3 = 1$ (with algebraic multiplicity 2). Since $\mathrm{Re}(\lambda_1) = \mathrm{Re}(\lambda_2) < 0$, these eigenvalues contribute to $E^s(\mathbf{0})$. Since $\lambda_3 > 1$, this eigenvalue contributes to $E^u(\mathbf{0})$. Indeed we can already conclude that $E^s(\mathbf{0})$ and $E^u(\mathbf{0})$ are two-dimensional and $E^c(\mathbf{0}) = \varnothing$.

An eigenvector for λ_1 is $(2, 2, 2, 1+\mathrm{i})^{\mathsf{T}}$. By taking its real and imaginary parts we obtain
$$E^s(\mathbf{0}) = \mathrm{span}\left(\left\{\begin{bmatrix} 2 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}\right).$$

The geometric multiplicity of λ_3 is 2, so we look at
$$(\lambda_3 I - A)^2 = \begin{bmatrix} 0 & 0 & 7 & -8 \\ 0 & 0 & 7 & -8 \\ 0 & 0 & 7 & -8 \\ 0 & 0 & 4 & -1 \end{bmatrix}.$$

The two left-most columns of this matrix are both $\mathbf{0}$, thus the first two standard basis vectors form a basis for its nullspace. Hence
$$E^u(\mathbf{0}) = \mathrm{span}\left(\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right\}\right).$$

References

[1] J.D. Meiss. *Differential Dynamical Systems.* SIAM, Philadelphia, 2007.

[2] C. Moler and C. Van Loan. Nineteen dubious ways to compute the exponential of a matrix, twenty-five years later. *SIAM Rev.*, 45(1):3–49, 2003.