Notes for 160.734
Part II: Existence and Uniqueness

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Before we study the behaviour of solutions to nonlinear ODEs, we first need to know when solutions exist and are unique. The main result is the Picard-Lindelöf theorem which tells us that there exists a unique solution if the ODEs are Lipschitz. The results require some familiarity of real analysis and closely follow chapter 3 of Meiss [1].

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1 The Picard-Lindelöf theorem

Example 1.1. Consider the one-dimensional ODE

$$\dot{x} = x^{\frac{1}{3}}. \tag{1.1}$$

This ODE is separable (meaning we can write $\dot{x} = g(x)h(t)$, for some functions g and h), and so we can directly integrate to obtain

$$\int x^{-\frac{1}{3}}\,dx = \int dt,$$
$$\frac{3}{2}x^{\frac{2}{3}} = t + C,$$

where $C \in \mathbb{R}$. With the general initial condition $x(t_0) = x_0$, we have $C = \frac{3}{2}x_0^{\frac{2}{3}} - t_0$, and so

$$x(t) = \left(x_0^{\frac{2}{3}} + \tfrac{2}{3}(t - t_0)\right)^{\frac{3}{2}}. \tag{1.2}$$

Now consider (1.1) with the initial condition $x(0) = 0$. Here $t_0 = 0$ and $x_0 = 0$ and (1.2) gives $x(t) = \left(\frac{2}{3}t\right)^{\frac{3}{2}}$. But this is not the only solution! Notice that $x(t) = 0$ is also a solution to the initial value problem. Indeed there is an uncountable family of solutions: $x(t)$ may take the value 0 until some time $\tau > 0$ at which it lifts off according to (1.2):

$$x(t) = \begin{cases} 0, & t \le \tau, \\ \left(\frac{2}{3}(t-\tau)\right)^{\frac{3}{2}}, & t \ge \tau. \end{cases}$$

- More generally we are concerned with the existence and uniqueness of solutions to the initial value problem

$$\dot{\mathbf{x}} = f(\mathbf{x}), \qquad \mathbf{x}(t_0) = \mathbf{x}_0, \tag{1.3}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$. The previous example shows that if f is non-differentiable, then (1.3) may not have a unique solution. It can be shown that if f is differentiable, then (1.3) has a unique solution. But we can do better!

Definition 1.1. Let $\mathcal{D} \subset \mathbb{R}^n$ be open. A function $f : \mathcal{D} \to \mathbb{R}^n$ is said to be *Lipschitz* if there exists $K \in \mathbb{R}$ such that

$$\|f(\mathbf{x}) - f(\mathbf{y})\| \le K\|\mathbf{x} - \mathbf{y}\|, \quad \text{for all } \mathbf{x}, \mathbf{y} \in \mathcal{D}. \tag{1.4}$$

- Every Lipschitz function is continuous (and also uniformly continuous).
- Every differentiable function $f : \mathcal{D} \to \mathbb{R}^n$ is Lipschitz on bounded subsets of $\mathcal{D}$.

Theorem 1.1 (Picard-Lindelöf)**.** *Consider the initial value problem* (1.3). *Let $b > 0$ and let*

$$\overline{B}_b(\mathbf{x}_0) = \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x} - \mathbf{x}_0\| \le b\}, \tag{1.5}$$

be the closed ball of radius b centred at $\mathbf{x}_0$. Suppose f is Lipschitz, for some $K \in \mathbb{R}$, in $\overline{B}_b(\mathbf{x}_0)$. Then (1.3) *has a unique solution for $t \in \left[t_0 - \frac{b}{M}, t_0 + \frac{b}{M}\right]$, where*

$$M = \max_{\mathbf{x} \in \overline{B}_b(\mathbf{x}_0)} \|f(\mathbf{x})\|. \tag{1.6}$$

- The Picard-Lindelöf theorem only allows for times within $\frac{b}{M}$ of t_0. This can be understood quite easily. The solution $\mathbf{x}(t)$ has a "speed" $|\dot{\mathbf{x}}(t)|$ of at most M and so in the worst case scenario could reach the boundary of $\overline{B}_b(\mathbf{x}_0)$ in a time $\frac{b}{M}$ (using speed is distance divided by time). Outside $\overline{B}_b(\mathbf{x}_0)$ we have no information on f and so we cannot say anything about the solution for times t with $|t - t_0| > \frac{b}{M}$.

Exercise 1.1. Consider

$$\dot{x} = x^2, \qquad x(0) = x_0\,, \tag{1.7}$$

where $x_0 > 0$.

i) Apply the Picard-Lindelöf theorem with an arbitrary value $b > 0$ so show that (1.7) has a unique solution for $|t| \leq \frac{b}{(x_0+b)^2}$ for any $b > 0$.

ii) Determine the value of b that maximises $\frac{b}{(x_0+b)^2}$ to show that (1.7) has a unique solution for $|t| \leq \frac{1}{4x_0}$.

iii) Solve (1.7) exactly to show that it has a unique solution for $t < \frac{1}{x_0}$. Thus the upper bound on t generated by the Picard-Lindelöf theorem is $\frac{1}{4}$ of the value of the actual upper bound.

2 The contracting mapping theorem

To prove the Picard-Lindelöf theorem we use the contraction mapping theorem. In this section we first state some important definitions, many of which should be familiar to you from real analysis, then state and prove the contraction mapping theorem (sometimes called the Banach fixed point theorem). In the next section we prove the Picard-Lindelöf theorem.

- Please refer to standard real analysis, functional analysis, or abstract algebra texts (or just Wikipedia) for definitions of the following italicised terms.
- A *metric space* (X, ρ) is a *vector space* X with a *metric* ρ. A *normed vector space* $(X, \|\cdot\|)$ is a vector space X with a *norm* $\|\cdot\|$. An *inner product space* $(X, <\cdot,\cdot>)$ is a vector space X with an *inner product* $<\cdot,\cdot>$.
- Given an inner product $<\cdot,\cdot>$, we can define a norm by $\|u\| = \sqrt{<u,u>}$. Given a norm $\|\cdot\|$, we can define a metric by $\rho(u,v) = \|u - v\|$. In this way every inner product space is a normed vector space, and every normed vector space is a metric space.

Definition 2.1. A sequence $\{u_n\}_{n\geq 0}$ in (X, ρ) is said to *converge* if there exists $u \in X$ such that for all $\varepsilon > 0$ there exists $N \in \mathbb{Z}$ such that $\rho(u_n, u) < \varepsilon$ for all $n \geq N$.

Definition 2.2. A sequence $\{u_n\}_{n\geq 0}$ in (X, ρ) is called *Cauchy* if for all $\varepsilon > 0$ there exists $N \in \mathbb{Z}$ such that $\rho(u_m, u_n) < \varepsilon$ for all $m, n \geq N$.

Exercise 2.1. Show that every convergent sequence is Cauchy.

Definition 2.3. A metric space is said to be *complete* if every Cauchy sequence converges.

- A complete inner product space is called a *Hilbert space.* A complete normed vector space is called a *Banach space.* A complete metric space, well, it's just called a *complete metric space.*

Definition 2.4. A *fixed point* of a continuous function $T : X \to X$ is a point u^* for which $T(u^*) = u^*$.

Definition 2.5. Let (X, ρ) be a metric space. A function $T : X \to X$ is said to be a *contraction* if there exists $0 \leq c < 1$ such that

$$\rho(T(u), T(v)) \leq c\rho(u, v), \quad \text{for all } u, v \in X. \tag{2.1}$$

- Note that contractions are Lipschitz (with constant $K = c$), and hence also continuous.

Theorem 2.1 (Contraction mapping theorem)**.** *Let (X, ρ) be a complete metric space. Suppose $T : X \to X$ is a contraction. Then T has a unique fixed point.*

Proof. Here we prove not only that T has a unique fixed point u^*, but that $T^n(u_0) \to u^*$ as $n \to \infty$ for every $u_0 \in X$.

[Existence]. Choose any $u_0 \in X$ and define a sequence $\{u_n\}_{i\geq 0}$ by $u_i = T(u_{i-1})$ for each $i \geq 1$. Since T is a contraction, for any $n \geq 1$

$$\begin{aligned}
\rho(u_{n+1}, u_n) &= \rho(T(u_n), T(u_{n-1})) \\
&\leq c\rho(u_n, u_{n-1}) = c\rho(T(u_{n-1}), T(u_{n-2})) \\
&\leq c^2\rho(u_{n-1}, u_{n-2}) \\
&\;\vdots \\
&\leq c^n\rho(u_1, u_0).
\end{aligned}$$

Thus for any $m > n$, by the triangle inequality

$$\rho(u_m, u_n) \leq \sum_{i=n}^{m-1} \rho(u_{i+1}, u_i) \leq \sum_{i=n}^{m-1} c^i \rho(u_1, u_0).$$

Using the formula for the sum of a truncated geometric series[1]

$$\rho(u_m, u_n) \leq \frac{c^n \rho(u_1, u_0)(1 - c^{m-n})}{1 - c}$$
$$\leq \frac{c^n \rho(u_1, u_0)}{1 - c}.$$

Since $c < 1$, we have $\rho(u_m, u_n) \to 0$ as $n \to \infty$. That is, $\{u_n\}$ is a Cauchy sequence. Since (X, ρ) is complete, $\{u_n\}$ converges to some $u^* \in X$. Then u^* is a fixed point of T because

$$T(u^*) = T\Big(\lim_{n\to\infty} u_n\Big) = \lim_{n\to\infty} T(u_n)$$
$$= \lim_{n\to\infty} u_{n+1} = u^*,$$

where we are permitted to swap T and the limit because T is continuous.

[Uniqueness]. Let $v^* \in X$ be another fixed point. Then

$$\rho(u^*, v^*) = \rho(T(u^*), T(v^*)) \leq c\rho(u^*, v^*).$$

But $0 \leq c < 1$, thus $\rho(u^*, v^*) = 0$, that is, $u^* = v^*$. □

3 Proof of the Picard-Lindelöf theorem

Proof of Theorem 1.1. In order to apply the contraction mapping theorem we first need to identify a suitable complete metric space. Let

$$V = C^0\left(\left[t_0 - \frac{b}{M}, t_0 + \frac{b}{M}\right], \overline{B}_b(\mathbf{x}_0)\right), \qquad (3.1)$$

be the set of all continuous functions

$$u : \left[t_0 - \frac{b}{M}, t_0 + \frac{b}{M}\right] \to \overline{B}_b(\mathbf{x}_0).$$

We first work with the ∞-norm, $\|u\|_\infty = \sup_t \|u(t)\|$. Then with the induced metric $\rho(u, v) = \|u - v\|_\infty$, (V, ρ) is a complete metric space.

We now define a function T on V as follows. For any $u = u(t) \in V$ let

$$T(u(t)) = \mathbf{x}_0 + \int_{t_0}^{t} f(u(s))\, ds. \qquad (3.2)$$

Notice that if $u^* = u^*(t)$ is a fixed point of T, then $u^*(t) = \mathbf{x}_0 + \int_{t_0}^t f(u^*(s))\, ds$, and differentiating this with respect to t gives $\dot{u}^*(t) = f(u^*(t))$. Also $u^*(t_0) = \mathbf{x}_0$, thus $u^*(t)$ would be a solution to the initial value problem (1.3). In view of the contraction mapping theorem it remains for us to show that $T : V \to V$ and that T is a contraction.

For any $u \in V$ and $t \in \left[t_0, t_0 + \frac{b}{M}\right]$,

$$\begin{aligned}
\|T(u(t)) - \mathbf{x}_0\| &= \left\| \int_{t_0}^{t} f(u(s))\, ds \right\| \\
&\leq \int_{t_0}^{t} \|f(u(s))\|\, ds \\
&\leq \int_{t_0}^{t} M\, ds \\
&= (t - t_0)M \\
&\leq b,
\end{aligned}$$

where $\|f(u(s))\| < M$ because $u(s) \in \overline{B}_b(\mathbf{x}_0)$ for all s. Similarly, $\|T(u(t)) - \mathbf{x}_0\| \leq b$ for all $t \in \left[t_0 - \frac{b}{M}, t_0\right]$. Thus $T(u(t)) \in \overline{B}_b(\mathbf{x}_0)$. Also $T(u(t))$ is a continuous function in view of the way T is defined. Thus T is indeed a function from V to V.

For any $u, v \in V$,

$$\begin{aligned}
\rho(T(u), T(v)) &= \sup_t \|T(u(t)) - T(v(t))\| \\
&= \sup_t \left\| \int_{t_0}^{t} f(u(s)) - f(v(s))\, ds \right\|.
\end{aligned}$$

Then

$$\begin{aligned}
&\sup_{t \in [t_0, t_0 + \frac{b}{M}]} \int_{t_0}^{t} \|f(u(s)) - f(v(s))\|\, ds \\
&\leq \sup_{t \in [t_0, t_0 + \frac{b}{M}]} \int_{t_0}^{t} K\|u(s) - v(s)\|\, ds \\
&\leq \sup_{t \in [t_0, t_0 + \frac{b}{M}]} \int_{t_0}^{t} K\|u - v\|_\infty\, ds \\
&= \sup_{t \in [t_0, t_0 + \frac{b}{M}]} K\|u - v\|_\infty (t - t_0) \\
&= \sup_{t \in [t_0, t_0 + \frac{b}{M}]} \frac{Kb}{M} \|u - v\|_\infty .
\end{aligned}$$

The same result holds for $t \in \left[t_0 - \frac{b}{M}, t_0\right]$, thus

$$\rho(T(u), T(v)) \leq \frac{Kb}{M} \rho(u, v).$$

[1] $\sum$ geo. series $= \frac{a(1-r^N)}{1-r}$, where a is the first term in the series, r is the ratio, and N is the number of terms.

Thus if $\frac{Kb}{M} < 1$ then T is a contraction and by the contraction mapping theorem has a unique fixed point u^* which, as noted above, is the solution to (1.3) that we require. Unfortunately, we may have $\frac{Kb}{M} \geq 1$.

To resolve this issue we consider instead the Bielecki norm (see [1], page 89),

$$\|u\|_L = \sup_t \mathrm{e}^{-L|t-t_0|}\|u(t)\|,$$

where $L > 0$. By repeating the above steps with the Bielecki norm we obtain $\frac{Kb}{M} < 1$ as long as $L \geq K$ (we omit these calculations for brevity). $\square$

4 Generalisations and extensions

- Intuitively, if (1.3) has a unique solution then we would expect it to have a unique solution for any other initial condition $\mathbf{y}$ sufficiently close to $\mathbf{x}_0$, and that the solution varies smoothly with respect to $\mathbf{y}$. The next theorem formalises this remark.

Theorem 4.1. *Consider the initial value problem (1.3). Let $b > 0$ and suppose f is Lipschitz, for some $K \in \mathbb{R}$, in $\overline{B}_b(\mathbf{x}_0)$. Let $M = \max_{\mathbf{x}\in\overline{B}_b(\mathbf{x}_0)} \|f(\mathbf{x})\|$ and let $a = \frac{b}{2M}$. Then for all $\mathbf{y} \in \overline{B}_{\frac{b}{2}}(\mathbf{x}_0)$, the initial value problem*

$$\dot{\mathbf{x}} = f(\mathbf{x}), \qquad \mathbf{x}(t_0) = \mathbf{y}, \tag{4.1}$$

has a unique solution for $t \in [t_0 - a, t_0 + a]$, call it $u(t;\mathbf{y})$. Moreover, $u(t;\mathbf{y})$ is uniformly Lipschitz[2] *in $\mathbf{y}$ with Lipschitz constant e^{Ka}.*

- The existence and uniqueness of $u(t;\mathbf{y})$ can be proved in the same way as for the Picard-Lindelöf theorem, and so we omit a proof.
- To prove that $u(t;\mathbf{y})$ is uniformly Lipschitz we use differential inequalities. If $g(t)$ is differentiable with $\dot{g}(t) \leq Kg(t)$ on $[t_0, t_0 + a]$, then $g(t)$ is bounded by the solution to $\dot{x} = Kx$ with the same initial condition. The following result generalises this to allow functions g that are only continuous.

Theorem 4.2 (Grönwall's inequality)**.** *Let $t_0 \in \mathbb{R}$ and $a > 0$. Let $g : [t_0, t_0 + a] \to \mathbb{R}$ be continuous. Let*

$$G(t) = g(t_0) + K\int_{t_0}^{t} g(s)\,ds, \tag{4.2}$$

where $K \geq 0$ and suppose that $g(t) \leq G(t)$ for all $t \in [t_0, t_0 + a]$. Then

$$g(t) \leq \mathrm{e}^{K(t-t_0)}g(t_0). \tag{4.3}$$

Proof. The function G is differentiable with $\dot{G}(t) = Kg(t)$, thus $\dot{G}(t) \leq KG(t)$. That is, $\dot{G}(t) - KG(t) \leq 0$. By multiplying this by its integrating factor e^{-Kt}, we can collect the two terms into a single derivative as

$$\frac{d}{dt}\left(\mathrm{e}^{-Kt}G(t)\right) \leq 0.$$

This tells us that the value of $\mathrm{e}^{-Kt}G(t)$ at any $t \geq t_0$ must be less than or equal to its value at $t = t_0$, i.e.

$$\mathrm{e}^{-Kt}G(t) \leq \mathrm{e}^{-Kt_0}G(t_0).$$

Then

$$g(t) \leq G(t) \leq \mathrm{e}^{K(t-t_0)}G(t_0) = \mathrm{e}^{K(t-t_0)}g(t_0),$$

as required. $\square$

Proof that $u(t;\mathbf{y})$ is uniformly Lipschitz. Choose any $\mathbf{y}, \mathbf{z} \in \overline{B}_{\frac{b}{2}}(\mathbf{x}_0)$. Define a function g by

$$g(t) = \|u(t;\mathbf{y}) - u(t;\mathbf{z})\|. \tag{4.4}$$

Since $u(t;\mathbf{y})$ is a solution to (4.1), it satisfies the following integral equation (obtained by integrating the ODE)

$$u(t;\mathbf{y}) = \mathbf{y} + \int_{t_0}^{t} f(u(s;\mathbf{y}))\,ds.$$

The same equation holds for $\mathbf{z}$, thus for all $t \in [t_0, t_0 + a]$

$$\begin{aligned} g(t) &= \left\|\mathbf{y} + \int_{t_0}^{t} f(u(s;\mathbf{y}))\,ds - \mathbf{z} - \int_{t_0}^{t} f(u(s;\mathbf{z}))\,ds\right\| \\ &\leq \|\mathbf{y} - \mathbf{z}\| + \int_{t_0}^{t} \|f(u(s;\mathbf{y})) - f(u(s;\mathbf{z}))\|\,ds \\ &\leq \|\mathbf{y} - \mathbf{z}\| + K\int_{t_0}^{t} \|u(s;\mathbf{y}) - u(s;\mathbf{z})\|\,ds \\ &= g(t_0) + K\int_{t_0}^{t} g(s)\,ds. \end{aligned}$$

[2] This means that for all $t \in \left[t_0 - \frac{b}{2M}, t_0 + \frac{b}{2M}\right]$ and all $\mathbf{y}, \mathbf{z} \in \overline{B}_{\frac{b}{2}}(\mathbf{x}_0)$, we have $\|u(t;\mathbf{y}) - u(t;\mathbf{z})\| \leq \mathrm{e}^{Ka}\|\mathbf{y} - \mathbf{z}\|$.

Then by Grönwall's inequality,

$$g(t) \le e^{K(t-t_0)} g(t_0) \le e^{Ka} g(t_0).$$

That is,

$$\|u(t;\mathbf{y}) - u(t;\mathbf{z})\| \le e^{Ka} \|\mathbf{y} - \mathbf{z}\|,$$

as required. □

- Many ODEs involve parameters. For example the angular displacement of a pendulum $\theta(t)$ may be well-modelled by

$$\begin{aligned}\dot{\theta} &= \phi, \\ \dot{\phi} &= \frac{-g}{\ell}\sin(\theta),\end{aligned}$$

where g (the acceleration due to gravity) and ℓ (the length of the pendulum) are parameters.

- Lastly we provide a result telling us that if we vary a parameter of f in a continuous fashion, then the solution to (1.3) also varies continuously. For a proof see [1], page 97.

Theorem 4.3. *Suppose* $f : \overline{B}_b(\mathbf{x}_0) \times \overline{B}_r(\mu_0) \to \mathbb{R}^n$ *has uniformly Lipschitz dependence on* $\mathbf{x} \in \overline{B}_b(\mathbf{x}_0)$ *and is a uniformly continuous function of parameters* $\mu \in \overline{B}_r(\mu_0)$*. Then for all* $\mathbf{y} \in \overline{B}_{\frac{b}{2}}(\mathbf{x}_0)$*, the initial value problem*

$$\dot{x} = f(\mathbf{x};\mu), \qquad \mathbf{x}(t_0) = \mathbf{y},$$

has a unique solution that is a uniformly continuous function of μ *on* $[t_0 - a, t_0 + a]$*, where* $a > 0$.

5 Bounds for ODEs on $\mathbb{R}$

Theorem 5.1. *Let* $g : \mathbb{R} \to \mathbb{R}$ *be locally Lipschitz. Let* $u : [t_0, t_0 + a] \to \mathbb{R}$ *and* $v : [t_0, t_0 + a] \to \mathbb{R}$ *be differentiable functions satisfying* $u(t_0) \le v(t_0)$ *and*

$$\dot{u}(t) \le g(u(t)), \qquad \dot{v}(t) = g(v(t)), \tag{5.1}$$

for all $t \in [t_0, t_0 + a]$*. Then* $u(t) \le v(t)$ *for all* $t \in [t_0, t_0 + a]$.

Proof. Suppose for a contradiction that $u(t) > v(t)$ for some $t \in [t_0, t_0 + a]$. Then in $[t_0, t_0 + a]$ there exists a last time t_1 for which $u(t) \le v(t)$. This means that $u(t_1) = v(t_1)$ and $u(t) > v(t)$ for all $t \in (t_1, t_2)$, for some t_2.

Let $w(t) = u(t) - v(t)$. For any $t \in (t_1, t_2)$,

$$\dot{w}(t) \le g(u(t)) - g(v(t)) \le K(u(t) - v(t)) = Kw(t),$$

where $K \ge 0$ is a Lipschitz constant for g, and $w(t_1) = 0$. Thus $w(t) \le 0$. That is $u(t) \le v(t)$, which is a contradiction. □

Example 5.1. Consider the initial value problem

$$\dot{x} = x^2 + \sin(3x), \qquad x(0) = x_0\,. \tag{5.2}$$

Given $x_0 > 0$ our goal is to find $\tau(x_0) > 0$ such that (5.2) has a solution for all $t \in [0, \tau(x_0))$ by using Theorem 5.1.

Let $f(x) = x^2 + \sin(3x)$ and let $u(t;x_0)$ be the solution to (5.2). Then $u(0,x_0) = x_0$ and $\dot{u}(t;x_0) = f(u(t;x_0))$ for $t \ge 0$ (really we mean for all $t \ge 0$ for which u is defined). Notice that $f(x) > 0$ for all $x > 0$ (because if $0 < x < \frac{\pi}{3}$, then $\sin(3x) > 0$ and so $f(x) > 0$, while if $x \ge \frac{\pi}{3}$ then $x^2 > 1$ and so $f(x) > 0$). Thus $u(t;x_0) > 0$ for $t \ge 0$, because $x_0 > 0$.

Let $g(x) = x^2 + 1$. Let $v(t;x_0)$ be the solution to the initial value problem

$$\dot{x} = x^2 + 1, \qquad x(0) = x_0\,. \tag{5.3}$$

Then $v(0,x_0) = x_0$ and $\dot{v}(t;x_0) = g(v(t;x_0))$ for $t \ge 0$. Since $f(x) \le g(x)$ for all $x > 0$, we can apply Theorem 5.1 to conclude that $u(t;x_0) \le v(t;x_0)$.

Now we solve (5.3) explicitly:

$$\begin{aligned}\int \frac{1}{x^2+1}\,dx &= \int dt \\ \tan^{-1}(x) &= t + C \\ \tan^{-1}(x) &= t + \tan^{-1}(x_0) \\ v(t;x_0) &= \tan\bigl(t + \tan^{-1}(x_0)\bigr).\end{aligned}$$

Thus $v(t;x_0)$ is well-defined until it blows up at $t = \frac{\pi}{2} - \tan^{-1}(x_0) = \cot^{-1}(x_0)$. Therefore $u(t;x_0)$ is well-defined for all $t \in [0, \tau(x_0))$, where $\tau(x_0) = \cot^{-1}(x_0)$.

References

[1] J.D. Meiss. *Differential Dynamical Systems.* SIAM, Philadelphia, 2007.