Notes for 160.734
Part III: Stability, Topological Equivalence, and Attractors

D.J.W. Simpson
SFS, Massey University
June 24, 2019

Here we begin to look at the qualitative properties of nonlinear ODEs. The most important result here is the Hartman-Grobman theorem which tells us that the dynamics near a hyperbolic equilibrium is topologically equivalent (and in fact conjugate) to its linearisation (which we can fully understand as it is a linear system). The technical terms mentioned here will be defined as we go along.

---

1 Preliminaries

- Let $\mathcal{X}$ be a manifold[1]. We can think of $\mathcal{X}$ as a smooth n-dimensional surface in $\mathbb{R}^k$, for some $k \geq n$ (often $\mathcal{X} = \mathbb{R}^n$).
- Let $f : \mathcal{X} \to \mathcal{X}$. Our interest here is in the behaviour of solutions to

$$\dot{\mathbf{x}} = f(\mathbf{x}). \tag{1.1}$$

- Previously we may have written the solution to (1.1), for a given initial condition $\mathbf{x}(0) = \mathbf{x}_0$, as $\mathbf{x}(t; \mathbf{x}_0)$, or just $\mathbf{x}(t)$, where t takes values in some allowed range. From now on we will write this solution as, say, $\varphi_t(\mathbf{x}_0)$, where, in particular, the introduction of a new symbol (here φ) helps remove a possible cause for confusion.
- Also, to simplify notation and to help us think more abstractly, we will usually write $\mathbf{x}$ instead of $\mathbf{x}_0$.
- For all $\mathbf{x} \in \mathcal{X}$ we have

$$\varphi_0(\mathbf{x}) = \mathbf{x}, \tag{1.2}$$

because the solution satisfies the initial condition. Also, for any times s and t we have

$$\varphi_t(\varphi_s(\mathbf{x})) = \varphi_{s+t}(\mathbf{x}), \tag{1.3}$$

because if we solve (1.1) from $\mathbf{x}$ for a time s, then an additional time t, the point at which we end up at must be the same as what we would get by solving (1.1) from $\mathbf{x}$ for a time $s + t$.

Definition 1.1.
The *forward orbit* of $\mathbf{x}$ is

$$\Gamma^+_{\mathbf{x}} = \{\varphi_t(\mathbf{x}) \mid t \geq 0\}.$$

The *backward orbit* of $\mathbf{x}$ is

$$\Gamma^-_{\mathbf{x}} = \{\varphi_t(\mathbf{x}) \mid t \leq 0\}.$$

The *orbit* of $\mathbf{x}$ is

$$\Gamma_{\mathbf{x}} = \Gamma^-_{\mathbf{x}} \cup \Gamma^+_{\mathbf{x}}.$$

Definition 1.2. Let $\Lambda \subset \mathcal{X}$. For any t, write

$$\varphi_t(\Lambda) = \{\varphi_t(\mathbf{x}) \mid \mathbf{x} \in \Lambda\}.$$

- Λ is a *forward invariant set* of (1.1) if $\varphi_t(\Lambda) \subset \Lambda$ for all $t > 0$.
- Λ is a *backward invariant set* of (1.1) if $\varphi_t(\Lambda) \subset \Lambda$ for all $t < 0$.
- Λ is an *invariant set* of (1.1) if $\varphi_t(\Lambda) \subset \Lambda$ for all t.

2 Linearisation

- First let us clarify how we describe the error terms of an asymptotic expansion (e.g. a Taylor expansion) of a function $g(\mathbf{x})$ about a point $\tilde{\mathbf{x}}$.

Definition 2.1. Let $g : \mathbb{R}^n \to \mathbb{R}^n$.

- The function g is said to be $\mathcal{O}\big((\mathbf{x} - \tilde{\mathbf{x}})^k\big)$ as $\mathbf{x} \to \tilde{\mathbf{x}}$ (or $\mathcal{O}(k)$ for short) if $\limsup_{\mathbf{x} \to \tilde{\mathbf{x}}} \frac{\|g(\mathbf{x})\|}{\big\|(\mathbf{x} - \tilde{\mathbf{x}})^k\big\|}$ is finite (this is "big-O notation").

[1] i.e. a topological space that resembles $\mathbb{R}^n$, for some n, near any point on $\mathcal{X}$.

- The function g is said to be $o\big((\mathbf{x}-\tilde{\mathbf{x}})^k\big)$ as $\mathbf{x} \to \tilde{\mathbf{x}}$ (or $o(k)$ for short) if $\lim_{\mathbf{x}\to\tilde{\mathbf{x}}} \frac{\|g(\mathbf{x})\|}{\|(\mathbf{x}-\tilde{\mathbf{x}})^k\|} = 0$ (this is "little-o notation").

Example 2.1. About $\tilde{x} = 0$, the function $g(x) = 7x^3+8x^4$ is $\mathcal{O}(3)$ (third order in x). It would also be correct (although perhaps not terribly helpful) to say that $g(x)$ is $\mathcal{O}(k)$ for any $k \leq 3$. Furthermore, $g(x)$ is $o(k)$ for any $k < 3$.

- Let $\mathbf{x}^* \in \mathcal{X}$ be an equilibrium of (1.1) (i.e. $f(\mathbf{x}^*) = \mathbf{0}$). Consider a Taylor expansion of f centred at $\mathbf{x}^*$. If f is C^1 we can write

$$f(\mathbf{x}) = Df(\mathbf{x}^*)(\mathbf{x}-\mathbf{x}^*) + o(1),$$

while if f is C^2 we can write

$$f(\mathbf{x}) = Df(\mathbf{x}^*)(\mathbf{x}-\mathbf{x}^*) + \mathcal{O}(2).$$

Note, $Df(\mathbf{x}^*)$ is the Jacobian of f evaluated at $\mathbf{x}^*$.

Definition 2.2. The *linearisation* of (1.1) about an equilibrium $\mathbf{x}^*$ is the ODE

$$\dot{\mathbf{x}} = Df(\mathbf{x}^*)(\mathbf{x}-\mathbf{x}^*). \tag{2.1}$$

- Equation (2.1) is linear and so we can understand the behaviour of its orbits completely. Indeed under the change of variables $\mathbf{y} = \mathbf{x}-\mathbf{x}^*$, (2.1) takes the familiar form $\dot{\mathbf{y}} = A\mathbf{y}$ where $A = Df(\mathbf{x}^*)$.
- The idea is that near $\mathbf{x}^*$ orbits of (1.1) behave similarly to orbits of (2.1). The accuracy and usefulness of (2.1) as an approximation to (1.1) near $\mathbf{x}^*$ will begin to be addressed in §4.

Definition 2.3. If $Df(\mathbf{x}^*)$ has no eigenvalues with zero real part, then $\mathbf{x}^*$ is said to be *hyperbolic*.

Definition 2.4. Let $\mathbf{x}^*$ be a hyperbolic equilibrium of (1.1).

- If all eigenvalues of $Df(\mathbf{x}^*)$ have negative real part, then $\mathbf{x}^*$ is said to be a *sink*.
- If all eigenvalues of $Df(\mathbf{x}^*)$ have positive real part, then $\mathbf{x}^*$ is said to be a *source*.
- Otherwise $\mathbf{x}^*$ is said to be a *saddle*.

Exercise 2.1. Classify each equilibrium of

$$\dot{x} = x^2 - y^2,$$
$$\dot{y} = x + 2y + 3,$$

as either a sink, a source, a saddle, or non-hyperbolic.

3 Stability

- We write

$$B_r(\mathbf{z}) = \{\mathbf{x} \in \mathcal{X} \mid \|\mathbf{x}-\mathbf{z}\| < r\}, \tag{3.1}$$

to denote the open ball centred at a point $\mathbf{z}$ with radius $r > 0$.

Definition 3.1. An equilibrium $\mathbf{x}^*$ of (1.1) is said to be *attracting* if there exists $\delta > 0$ such that $\varphi_t(\mathbf{x}) \to \mathbf{x}^*$ for all $\mathbf{x} \in B_\delta(\mathbf{x}^*)$.

Exercise 3.1. Consider the two-dimensional system given in polar coordinates[2] by

$$\begin{aligned} \dot{r} &= r(1-r), \\ \dot{\theta} &= \sin^2\left(\frac{\theta}{2}\right). \end{aligned} \tag{3.2}$$

i) Identify the two equilibria of (3.2).

ii) Sketch a phase portrait of (3.2) (i.e. draw a representative collection of orbits of (3.2) in the (x, y)-plane).

iii) Argue that one of the equilibria is attracting. Notice, however, that there exist points arbitrarily close to this equilibrium whose forward orbits travel far from the equilibrium (before eventually limiting to it asymptotically). For this reason this equilibrium is not what we wish to think of as a 'stable' equilibrium.

Definition 3.2. An equilibrium $\mathbf{x}^*$ of (1.1) is said to be *Lyapunov*[3] *stable* if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $\mathbf{x} \in B_\delta(\mathbf{x}^*)$ we have $\varphi_t(\mathbf{x}) \in B_\varepsilon(\mathbf{x}^*)$ for all $t \geq 0$.

- Roughly speaking an equilibrium is Lyapunov stable if forward orbits that start near $\mathbf{x}^*$ stay near $\mathbf{x}^*$.

[2] $x = r\cos(\theta)$, $y = r\sin(\theta)$

[3] Aleksandr Lyapunov (1857–1918) was a Russian mathematician who made many important contributions to the theory of dynamical systems that we will encounter.

Definition 3.3. An equilibrium $\mathbf{x}^*$ of (1.1) is said to be *asymptotically stable* if it is attracting and Lyapunov stable.

Theorem 3.1. *If $\mathbf{x}^*$ is a sink, then $\mathbf{x}^*$ is asymptotically stable.*

- A proof of Theorem 3.1 is given in [1] (pages 121–122). The proof uses variation of parameters and Grönwall's inequality.

Exercise 3.2. Show that $\dot{x} = -x^3$ provides a counterexample to the converse of Theorem 3.1.

- The asymptotic stability of equilibria of (1.1) is one of the most useful and important things we can say about the ODE system. Theorem 3.1 tells us that the asymptotically stability of an equilibrium $\mathbf{x}^*$ can be demonstrated by showing that all eigenvalues of $Df(\mathbf{x}^*)$ have negative real part. However, this requires that f is at least C^1 at $\mathbf{x}^*$. The next theorem provides a way of demonstrating asymptotically stability when f is only continuous.

Definition 3.4. Suppose f is continuous and $\mathbf{x}^*$ is an equilibrium of (1.1). A continuous function $L : \mathcal{X} \to \mathbb{R}$ is said to be a *Lyapunov function* for $\mathbf{x}^*$ if there exists $\delta > 0$ such that

i) $L(\mathbf{x}^*) = 0$,

ii) $L(\mathbf{x}) > 0$ for all $\mathbf{x} \in B_\delta(\mathbf{x}^*) \setminus \{\mathbf{x}^*\}$,

iii) $L(\varphi_t(\mathbf{x})) < L(\mathbf{x})$ for all $\mathbf{x} \in B_\delta(\mathbf{x}^*) \setminus \{\mathbf{x}^*\}$ and all $t > 0$.

Theorem 3.2. *If there exists a Lyapunov function for $\mathbf{x}^*$, then $\mathbf{x}^*$ is asymptotically stable.*

- For a proof of Theorem 3.2 see [1], pages 123–124.

Exercise 3.3. For $\dot{x} = -x^3$, find a Lyapunov function for the equilibrium 0.

- Finally we generalise the notion of stability to arbitrary invariant sets.

Definition 3.5. An invariant set Λ of (1.1) is said to be *Lyapunov stable* if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $\mathbf{x}$ within δ of Λ[4], we have that $\varphi_t(\mathbf{x})$ is within ε of Λ for all $t \geq 0$.

Definition 3.6. An invariant set Λ of (1.1) is said to be *asymptotically stable* if it is Lyapunov stable and there exists $\delta > 0$ such that for all $\mathbf{x}$ within δ of Λ, we have $\varphi_t(\mathbf{x}) \to \Lambda$ as $t \to \infty$[5].

4 Topological equivalence

- Let $f : \mathcal{X} \to \mathcal{X}$ and $g : \mathcal{Y} \to \mathcal{Y}$. Let $\varphi_t(\mathbf{x})$ and $\psi_t(\mathbf{y})$, respectively, denote the solutions to

$$\dot{\mathbf{x}} = f(\mathbf{x}), \tag{4.1}$$
$$\dot{\mathbf{y}} = g(\mathbf{y}). \tag{4.2}$$

- Here we introduce several ways, some stronger than others, by which we mean to say that the dynamics of (4.1) and (4.2) are the 'same'.

Definition 4.1. We say that (4.1) and (4.2) are *conjugate* if there exists a homeomorphism[6] $h : \mathcal{X} \to \mathcal{Y}$ such that

$$h(\varphi_t(\mathbf{x})) = \psi_t(h(\mathbf{x})), \tag{4.3}$$

for all $\mathbf{x} \in \mathcal{X}$ and all $t \in \mathbb{R}$.

Exercise 4.1. Use $h(x) = x^3$ to show that $\dot{x} = -x$ and $\dot{y} = -3y$ are conjugate.

- Let $\mathbf{x} \in \mathcal{X}$ and consider its orbit $\Gamma_\mathbf{x}$ under (4.1). Let $\mathbf{y} = h(\mathbf{x})$, then

$$\begin{aligned} h(\Gamma_\mathbf{x}) &= \{h(\varphi_t(\mathbf{x})) \mid t \in \mathbb{R}\} \\ &= \{\psi_t(\mathbf{y}) \mid t \in \mathbb{R}\}, \end{aligned}$$

  is the orbit of $\mathbf{y}$ for (4.2). Therefore under h every orbit of (4.1) maps to an orbit of (4.2).
- In particular, every equilibrium of (4.1) corresponds to an equilibrium of (4.2). Also every periodic orbit of (4.1) of period T corresponds to a periodic orbit of (4.2) of period T.
- Conjugate ODEs have the same 'temporal parameterisation'. This is quite restrictive and the next definition, which involves a weaker condition, is typically more useful.

[4] i.e. $\|\mathbf{x} - \mathbf{z}\| < \delta$ for some $\mathbf{z} \in \Lambda$.

[5] i.e. $\inf_{\mathbf{z} \in \Lambda} \|\varphi_t(\mathbf{x}) - \mathbf{z}\| \to 0$ as $t \to \infty$.

[6] A *homeomorphism* is a function that is one-to-one, onto, continuous, and has a continuous inverse.

Definition 4.2. We say that (4.1) and (4.2) are *topologically equivalent* if there exists a homeomorphism $h : \mathcal{X} \to \mathcal{Y}$ and a continuous function $\tau : \mathcal{X} \times \mathbb{R} \to \mathbb{R}$, that is increasing in t, such that

$$h\big(\varphi_{\tau(\mathbf{x},t)}(\mathbf{x})\big) = \psi_t(h(\mathbf{x})), \tag{4.4}$$

for all $\mathbf{x} \in \mathcal{X}$ and all $t \in \mathbb{R}$.

- Topologically equivalent ODEs exhibit the same orbit structure and direction of time.
- Sometimes we desire a stronger notion of equivalence than topological equivalence if we want differential properties of (4.1) and (4.2) to be preserved.

Definition 4.3. We say that (4.1) and (4.2) are *diffeomorphic* if there exists a diffeomorphism[7] $h : \mathcal{X} \to \mathcal{Y}$ such that (4.3) is satisfied for all $\mathbf{x} \in \mathcal{X}$ and all $t \in \mathbb{R}$.

Definition 4.4. We say that (4.1) and (4.2) are *smoothly equivalent* if there exists a diffeomorphism $h : \mathcal{X} \to \mathcal{Y}$ and a C^1 function $\tau : \mathcal{X} \times \mathbb{R} \to \mathbb{R}$, that is increasing in t, such that (4.4) is satisfied for all $\mathbf{x} \in \mathcal{X}$ and all $t \in \mathbb{R}$.

Theorem 4.1. *Let $f(\mathbf{x}) = A\mathbf{x}$ and $g(\mathbf{y}) = B\mathbf{y}$. Then (4.1) and (4.2) are diffeomorphic if and only if A and B are similar.*

Proof. First, suppose A and B are similar. Then $PA = BP$, for some nonsingular matrix P. Let $h(\mathbf{x}) = P\mathbf{x}$ (a diffeomorphism). Then

$$\begin{aligned} h(\varphi_t(\mathbf{x})) &= P\varphi_t(\mathbf{x}) \\ &= P\mathrm{e}^{tA}\mathbf{x} \\ &= \mathrm{e}^{tB}P\mathbf{x} \\ &= \psi_t(h(\mathbf{x})). \end{aligned}$$

Hence (4.1) and (4.2) are diffeomorphic.

Conversely, suppose (4.1) and (4.2) are diffeomorphic. Then (4.3) gives us

$$h\big(\mathrm{e}^{tA}\mathbf{x}\big) = \mathrm{e}^{tB}h(\mathbf{x}).$$

By differentiating both sides with respect to $\mathbf{x}$ we obtain

$$Dh\big(\mathrm{e}^{tA}\mathbf{x}\big)\mathrm{e}^{tA} = \mathrm{e}^{tB}Dh(\mathbf{x}).$$

By putting $\mathbf{x} = \mathbf{0}$ and letting $P = Dh(\mathbf{0})$ (a nonsingular matrix) we produce

$$P\mathrm{e}^{tA} = \mathrm{e}^{tB}P.$$

By differentiating both sides with respect to t, then substituting $t = 0$, we obtain $PA = BP$, as required. □

Exercise 4.2. Here you will consider the equivalence of (4.1) and (4.2) with $f(\mathbf{x}) = A\mathbf{x}$ and $g(\mathbf{y}) = B\mathbf{y}$, where

$$A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}, \qquad B = \begin{bmatrix} \lambda & \alpha \\ 0 & \lambda \end{bmatrix}. \tag{4.5}$$

Assume $\lambda \neq 0$ (so that $\mathbf{0}$ is hyperbolic) and $\alpha \neq 0$ (so that the eigenspace of B is only one-dimensional).

i) Use Theorem 4.1 to show that (4.1) and (4.2) are not diffeomorphic.

Define $h : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$h(x_1, x_2) = \begin{bmatrix} x_1 + \frac{\alpha}{\lambda} x_2 \ln|x_2| \\ x_2 \end{bmatrix},$$

where we take the value of $x_2 \ln|x_2|$ at $x_2 = 0$ to be 0.

ii) Show that h is a homeomorphism (Hint: construct h^{-1}) but not a diffeomorphism.

iii) Use h to show that (4.1) and (4.2) are conjugate (this is also a consequence of the following theorem).

Theorem 4.2. *Let $f(\mathbf{x}) = A\mathbf{x}$ and $g(\mathbf{y}) = B\mathbf{y}$. Suppose that $\mathbf{0}$ is a hyperbolic equilibrium of both (4.1) and (4.2). Let n_A and n_B be the number of eigenvalues (counting algebraic multiplicity) of A and B, respectively, that have negative real part. Then (4.1) and (4.2) are conjugate if and only if $n_A = n_B$.*

- The proof of Theorem 4.2 requires quite a lot of work and so we will skip it, see [1, 2].
- We now come to the most important result of this section. A proof (quite involved) is given in [2][8].

[7] A *diffeomorphism* is a function that is one-to-one, onto, C^1, and has a C^1 inverse.

[8] This result was first proved, independently, by Russian mathematician D.M. Grobman in 1959 and American mathematician Philip Hartman (1915–2015) in 1960 (not to be confused with Phil Hartman who does the voice for Troy McClure).

Theorem 4.3 (Hartman-Grobman)**.** *Let* $f : \mathcal{X} \to \mathcal{X}$ *be* C^1 *and let* $\mathbf{x}^*$ *be a hyperbolic equilibrium of* (4.1)*. Then there exists a neighbourhood of* $\mathbf{x}^*$ *within which* (4.1) *is conjugate to its linearisation* (2.1)*.*

- The change of variables $\mathbf{y} = h(\mathbf{x})$ given by the conjugacy of the Hartman-Grobman theorem may not be C^1. It is C^1 if the eigenvalues of $Df(\mathbf{x}^*)$ satisfy a surprisingly complicated non-resonance condition as given by Sternberg's linearisation theorem[9]. Here is a simple two-dimensional version of this theorem, see [4, 5].

Theorem 4.4. *Suppose* $f : \mathbb{R}^2 \to \mathbb{R}^2$ *is* C^∞ *with* $f(\mathbf{0}) = \mathbf{0}$ *and*

$$Df(\mathbf{0}) = \begin{bmatrix} \lambda & 0 \\ 0 & \sigma \end{bmatrix},$$

where $\lambda, \sigma \in \mathbb{R}$*. If* $p\lambda + q\sigma \neq 0$ *for every* $p, q \geq -1$ *with* $p + q \geq 1$*, then there exists a* C^∞ *change of variables under which* (1.1) *is transformed to* $\dot{\mathbf{x}} = Df(\mathbf{0})\mathbf{x}$ *in a neighbourhood of* $\mathbf{0}$*.*

- The necessity of the non-resonant condition can be seen by explicitly computing the change of variables through a two-dimensional Taylor expansion.

5 Attractors

Definition 5.1. A point $\mathbf{z} \in \mathcal{X}$ is a *limit point* of a continuous function $\phi : \mathbb{R} \to \mathcal{X}$ if there exists an increasing[10] sequence $\{t_k\}_{k=1}^\infty$, with $t_k \to \infty$ as $k \to \infty$, such that $\phi(t_k) \to \mathbf{z}$ as $k \to \infty$.

Definition 5.2. Let $\mathbf{x} \in \mathcal{X}$ and consider the forward orbit $\Gamma_{\mathbf{x}}^+$ of $\mathbf{x}$ under (1.1). The collection of all limit points of $\Gamma_{\mathbf{x}}^+$ is called the ω*-limit set* of $\mathbf{x}$, denoted $\omega(\mathbf{x})$.

Lemma 5.1. $\omega(\mathbf{x})$ *is closed*[11]*.*

Proof. Choose any $\mathbf{z} = \overline{\omega(\mathbf{x})}$ (the closure of $\omega(\mathbf{x})$). Then there exists a sequence $\{\mathbf{z}_n\}_{n=1}^\infty$ in $\omega(\mathbf{x})$ such that $\mathbf{z}_n \to \mathbf{z}$ as $n \to \infty$. For each n, there exists an increasing sequence of times $\{t_{n,k}\}_{k=1}^\infty$, with $t_{n,k} \to \infty$ as $k \to \infty$, such that $\varphi_{t_{n,k}}(\mathbf{x}) \to \mathbf{z}_n$ as $k \to \infty$.

For each n we can find k_n such that

$$\left\| \varphi_{t_{n,k_n}}(\mathbf{x}) - \mathbf{z}_n \right\| < \frac{1}{n}.$$

We can also choose the k_n such that $\{t_{n,k_n}\}_{n=1}^\infty$ is an increasing sequence with $t_{n,k_n} > n$ for each n. Then $\varphi_{t_{n,k_n}}(\mathbf{x}) \to \mathbf{z}$ and $t_{n,k_n} \to \infty$ as $n \to \infty$. Thus $\mathbf{z} \in \omega(\mathbf{x})$, and hence $\omega(\mathbf{x})$ is closed. □

Lemma 5.2. $\omega(\mathbf{x})$ *is an invariant of* (1.1)*.*

Proof. Choose any $\mathbf{z} \in \omega(\mathbf{x})$. Then there exists an increasing sequence $\{t_k\}_{k=1}^\infty$, with $t_k \to \infty$ as $k \to \infty$, such that $\varphi_{t_k}(\mathbf{x}) \to \mathbf{z}$ as $k \to \infty$.

Choose any $t \in \mathbb{R}$. Then $\{t_k + t\}_{k=1}^\infty$ is an increasing sequence tending to ∞ and $\varphi_{t_k+t}(\mathbf{x}) = \varphi_t(\varphi_{t_k}(\mathbf{x})) \to \varphi_t(\mathbf{z})$ as $k \to \infty$. Thus $\varphi_t(\mathbf{z}) \in \omega(\mathbf{x})$, and so $\omega(\mathbf{x})$ is an invariant of (1.1). □

- For a proof of the next result see [1].

Lemma 5.3. *If* $\Gamma_{\mathbf{x}}^+$ *is contained in a compact set, then* $\omega(\mathbf{x})$ *is non-empty, compact, and connected*[12]*.*

Exercise 5.1. A point $\mathbf{z} \in \mathcal{X}$ is said to be *non-wandering* if for all $\varepsilon > 0$ and all $t \in \mathbb{R}$, there exists $s \geq t$ such that $\varphi_s(B_\varepsilon(\mathbf{z})) \cap B_\varepsilon(\mathbf{z}) \neq \varnothing$.

Show that every $\mathbf{z} \in \omega(\mathbf{x})$ is non-wandering.

Exercise 5.2. A set $\Omega \subset \mathcal{X}$ is said to be *minimal* if it is closed, non-empty, invariant under (1.1), and does not contain any such set as a proper subset.

Show that a compact set Ω is minimal if and only if $\Omega = \omega(\mathbf{x})$ for some $\mathbf{x} \in \Omega$.

Theorem 5.4 (Poincaré-Bendixson)**.** *Suppose* $\mathcal{X}$ *is a simply connected subset*[13] *of* $\mathbb{R}^2$*. Let* $\mathbf{x} \in \mathcal{X}$ *and suppose* $\omega(\mathbf{x})$ *is contained in a compact set. If* $\omega(\mathbf{x})$ *contains no equilibria, then it is a periodic orbit.*

- The importance of the Poincaré-Bendixson theorem is that it tells us that chaos (which we will come to soon) requires a phase space of at least three dimensions.

[9] This theorem was first obtained in a general form in [3] by American mathematician Shlomo Sternberg (1936–).

[10] i.e. $t_1 < t_2 < \cdots$.

[11] A set is closed if and only if it contains all its limit points. Take care to note that the limit points of a set are different to the limit points of a function of t (Definition 5.1).

[12] A set $\Omega \subset \mathcal{X}$ is *connected* if it not contained within the disjoint union of two or more non-empty open subsets of $\mathcal{X}$.

[13] Simply connectedness is a strong form of connectedness basically meaning that the set has no holes (just Google it).

- The proof of the Poincaré-Bendixson theorem requires a series of topological arguments that we will skip, see [1] pages 220–222.

Exercise 5.3. Describe an instance of $\omega(\mathbf{x})$ that satisfies the conditions of Theorem 5.4 and is neither an equilibrium nor a periodic orbit. HINT: by Theorem 5.4, $\omega(\mathbf{x})$ must contain an equilibrium (but not *be* an equilibrium).

Definition 5.3. A compact set $\Omega \subset \mathcal{X}$ is said to be a *trapping region* for (1.1) if $\varphi_t(\Omega) \subset \mathrm{int}(\Omega)$[14] for all $t > 0$.

- Roughly speaking, Ω is a trapping region if $f(\mathbf{x})$ points 'inwards' for every $\mathbf{x}$ on the boundary of Ω. For this reason trapping regions are often quite easy to identify.

Definition 5.4. A set $\Lambda \subset \mathcal{X}$ is said to be an *attracting set* of (1.1) if there exists a trapping region Ω such that

$$\Lambda = \bigcap_{t \geq 0} \varphi_t(\Omega). \qquad (5.1)$$

Theorem 5.5. *Attracting sets are invariant.*

Proof. Let Λ be an attracting set of (1.1) and let Ω be a trapping region so that (5.1) holds. Choose any $\mathbf{x} \in \Lambda$ and any $s \in \mathbb{R}$ (it remains for us to show that $\varphi_s(\mathbf{x}) \in \Lambda$).

Observe $\varphi_t(\mathbf{x}) \in \Omega$ for all $t \leq 0$ (because $\mathbf{x} \in \Lambda$ and by (5.1)). Also $\varphi_t(\mathbf{x}) \in \Omega$ for all $t > 0$ because Ω is forward invariant. By substituting $t = \tilde{t} + s$ we have that $\varphi_{\tilde{t}+s}(\mathbf{x}) \in \Omega$ for all $\tilde{t} \geq 0$ (actually for all $\tilde{t} \in \mathbb{R}$ but this is not needed). That is $\varphi_{\tilde{t}}(\varphi_s(\mathbf{x})) \in \Omega$ for all $\tilde{t} \geq 0$, hence $\varphi_s(\mathbf{x}) \in \Lambda$ by (5.1). □

- For a proof of the next result see [1], page 147.

Theorem 5.6.

i) Every attracting set is asymptotically stable.

ii) Every compact asymptotically stable set is an attracting set.

- We are now ready to define the notion of an 'attractor'. We wish attractors to (i) be invariant, (ii) attract all nearby orbits, and (iii) be compact. Attracting sets satisfy all of these properties.
- However, we also wish attractors to be (iv) irreducible or indivisible, loosely meaning that they cannot be decomposed into 'sub-attractors'. Attractors are therefore defined as attracting sets with some irreducibility constraint. However, to date there is no universal agreement as to what this constraint should be! Some authors require an attractor to be an ω-limit set [1], be minimal (see Exercise 5.2) [6], or be 'chain-transitive' [2]. In contrast Milnor [7] defines an attractor merely as an invariant set that attracts a set of positive measure (we will come to measure theory in Part VIII).
- The different definitions have different strengths and weaknesses regarding what can be proved about attractors. For this course such differences will not be important; for concreteness we will work the following definition (given in, for instance, [8]):

Definition 5.5. An *attractor* is an attracting set that contains a dense orbit.

References

[1] J.D. Meiss. *Differential Dynamical Systems.* SIAM, Philadelphia, 2007.

[2] C. Robinson. *Dynamical Systems. Stability, Symbolic Dynamics, and Chaos.* CRC Press, Boca Raton, FL, 1999.

[3] S. Sternberg. On the structure of local homeomorphisms of Euclidean n-space, II. *Amer. J. Math.*, 80(3):623–631, 1958.

[4] R.C. Robinson. *An Introduction to Dynamical Systems. Continuous and Discrete.* Prentice Hall, Upper Saddle River, NJ, 2004.

[5] Z. Zhi-Fen, D. Tong-Ren, H. Wen-Zao, and D. Zhen-Xi. *Qualitative Theory of Differential Equations.*, volume 101 of *Translations of Mathematical Monographs.* American Mathematical Society, Providence, RI, 2006.

[6] L. Perko. *Differential Equations and Dynamical Systems.* Springer-Verlag, New York, 2001.

[7] J. Milnor. On the concept of attractor. *Commun. Math. Phys.*, 99:177–195, 1985.

[14] The *interior* of Ω.

[8] J. Guckenheimer and P.J. Holmes. *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields.* Springer-Verlag, New York, 1986.