Notes for 160.734
Part IV: Invariant Manifolds

D.J.W. Simpson
SFS, Massey University
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Attractors were introduced at end of the last section. If we can identify all the attractors of a system, determine their basins (defined below), and a few other essential properties (e.g. rate of convergence), then we should have a pretty good understanding of the long-time behaviour of system. Here we will look at invariant manifolds and use them to help us understand attractors.

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1 Stable and unstable manifolds

- As before we consider
$$\dot{\mathbf{x}} = f(\mathbf{x}), \tag{1.1}$$
where $f : \mathcal{X} \to \mathcal{X}$ and $\mathcal{X}$ is an n-dimensional manifold, and let $\varphi_t(\mathbf{x})$ denote the solution to (1.1).

Definition 1.1. Let Λ be an invariant set of (1.1). The *stable manifold* of Λ is
$$W^s(\Lambda) = \{\mathbf{x} \in \mathcal{X} \setminus \Lambda \mid \varphi_t(\mathbf{x}) \to \Lambda \text{ as } t \to \infty\}.$$
The *unstable manifold* of Λ is
$$W^u(\Lambda) = \{\mathbf{x} \in \mathcal{X} \setminus \Lambda \mid \varphi_t(\mathbf{x}) \to \Lambda \text{ as } t \to -\infty\}.$$

- It can be shown that $W^s(\Lambda)$ and $W^u(\Lambda)$ are actually manifolds, see pages 181–185 of [1].

Definition 1.2. If Λ is an attracting set then $W^s(\Lambda)$ is also known as the *basin of attraction* of Λ.

Lemma 1.1. *$W^s(\Lambda)$ and $W^u(\Lambda)$ are invariant.*

Proof. Choose any $\mathbf{z} \in W^s(\Lambda)$ and any $s \in \mathbb{R}$. Since $\varphi_t(\mathbf{z}) \to \Lambda$ as $t \to \infty$, we have that $\varphi_s(\varphi_t(\mathbf{z})) \to \varphi_s(\Lambda) = \Lambda$ as $t \to \infty$. But $\varphi_s(\varphi_t(\mathbf{z})) = \varphi_t(\varphi_s(\mathbf{z}))$. Therefore $\varphi_t(\varphi_s(\mathbf{z})) \to \Lambda$ as $t \to \infty$. Thus $\varphi_s(\mathbf{z}) \in W^s(\Lambda)$, hence $W^s(\Lambda)$ is invariant. The result for $W^u(\Lambda)$ can be shown similarly. $\square$

- As we will see later, intersections between stable and unstable manifolds provide one of the most fundamental mechanisms by which chaotic dynamics arises.

Definition 1.3. An orbit $\Gamma_{\mathbf{x}}$ is said to be *homoclinic* to an invariant set Λ if $\Gamma_{\mathbf{x}} \subset W^s(\Lambda) \cap W^u(\Lambda)$.

Definition 1.4. An orbit $\Gamma_{\mathbf{x}}$ is said to be *heteroclinic* to invariant sets Λ_1 and Λ_2 if $\Gamma_{\mathbf{x}} \subset W^s(\Lambda_1) \cap W^u(\Lambda_2)$.

- Another way to state Definition 1.4 is to say that $\varphi_t(\mathbf{x}) \to \Lambda_1$ as $t \to \infty$ and $\varphi_t(\mathbf{x}) \to \Lambda_2$ as $t \to -\infty$.

2 The stable manifold theorem

- The stable, unstable, and centre subspaces of an equilibrium of a linear system were defined in Part I. For an equilibrium $\mathbf{x}^*$ of (1.1), these subspaces are given by linearising about $\mathbf{x}^*$.

Example 2.1. Here we compute the stable, unstable, and centre subspaces of the equilibria of
$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = f(x,y) = \begin{bmatrix} -6xy \\ x - x^2 - y \end{bmatrix}. \tag{2.1}$$
The equilibria of (2.1) are $(0,0)$ and $(1,0)$.

i) First consider $(0,0)$. We have
$$Df(0,0) = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}.$$
This matrix has eigenvalues
$$\lambda_1 = -1, \qquad \lambda_2 = 0,$$
and corresponding eigenvectors
$$v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Therefore

$$\begin{aligned} E^s(0,0) &= \{cv_1 \mid c \in \mathbb{R}\}, \\ E^u(0,0) &= \varnothing, \\ E^c(0,0) &= \{cv_2 \mid c \in \mathbb{R}\}. \end{aligned}$$

ii) Second consider $(1,0)$. We have

$$Df(1,0) = \begin{bmatrix} 0 & -6 \\ -1 & -1 \end{bmatrix}.$$

This matrix has eigenvalues

$$\lambda_1 = -3, \qquad \lambda_2 = 2,$$

and corresponding eigenvectors

$$v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}.$$

Therefore

$$\begin{aligned} E^s(1,0) &= \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} + cv_1 \;\middle|\; c \in \mathbb{R} \right\}, \\ E^u(1,0) &= \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} + cv_2 \;\middle|\; c \in \mathbb{R} \right\}, \\ E^c(1,0) &= \varnothing. \end{aligned}$$

- The next result, which can be proved using the fundamental solution theorem, indicates that for linear ODEs the stable and unstable manifolds of an equilibrium are equal to its stable and unstable subspaces minus the equilibrium.

Lemma 2.1. *For the linear system* $\dot{\mathbf{x}} = A\mathbf{x}$, $W^s(\mathbf{0}) = E^s(\mathbf{0}) \setminus \{\mathbf{0}\}$ *and* $W^u(\mathbf{0}) = E^u(\mathbf{0}) \setminus \{\mathbf{0}\}$.

- Of course we more interested in nonlinear ODEs for which we have the following theorem.

Theorem 2.2 (Stable manifold theorem)**.** *Let* $\mathbf{x}^*$ *be a hyperbolic equilibrium of* (1.1) *and suppose* f *is* C^k *(*$k \geq 1$*) in a neighbourhood of* $\mathbf{x}^*$*. Then there exists* $\delta > 0$ *such that*

$$W^s_{\mathrm{loc}}(\mathbf{x}^*) = \big\{ \mathbf{x} \in W^s(\mathbf{x}^*) \;\big|\; \varphi_t(\mathbf{x}) \in B_\delta(\mathbf{x}^*) \text{ for all } t \geq 0 \big\}, \quad (2.2)$$

is a C^k *manifold and a graph over* E^s *that is tangent to* $E^s(\mathbf{x}^*)$ *at* $\mathbf{x}^*$*.*

- Theorem 2.2 tells us that, locally, $W^s(\mathbf{x}^*)$ is a manifold with the same dimension as $E^s(\mathbf{x}^*)$ and is tangent to $E^s(\mathbf{x}^*)$ as it emanates from $\mathbf{x}^*$.
- The same result can be seen to hold for $W^u(\mathbf{x}^*)$ by simply reversing the direction of time.

Example 2.2. Here we compute $W^s(1,0)$ to third order for (2.1).

Above we showed that $E^s(1,0)$ has slope $\frac{1}{2}$, thus by the stable manifold theorem $W^s(1,0)$ is a curve $y = \phi(x)$ where

$$\phi(x) = \frac{1}{2}(x-1) + a(x-1)^2 + b(x-1)^3 + \mathcal{O}\big((x-1)^4\big),$$

for some $a, b \in \mathbb{R}$. Our goal is to determine the values of a and b.

To make things easier for ourselves, we let $u = x - 1$. Then (2.1) becomes

$$\begin{bmatrix} \dot{u} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -6(1+u)y \\ -u - u^2 - y \end{bmatrix}. \quad (2.3)$$

We wish to compute the stable manifold of $(0,0)$ for (2.3). We write as $y = \psi(u)$, where $\psi(u) = \phi(u+1)$. Since stable manifolds are invariants, we must have $y(t) = \psi(u(t))$ for all t. Differentiating this with respect to t gives

$$\dot{y} = \psi'(u)\dot{u}. \quad (2.4)$$

The LHS of (2.4) is

$$\begin{aligned} \dot{y} &= -u - u^2 - \psi(u) \\ &= -\frac{3}{2}u - (1+a)u^2 - bu^3 + \mathcal{O}\big(u^4\big). \end{aligned} \quad (2.5)$$

The RHS of (2.4) is

$$\begin{aligned} \psi'(u)\dot{u} &= \left(\frac{1}{2} + 2au + 3bu^2 + \mathcal{O}\big(u^3\big) \right) \\ &\quad \times \big(-6(1+u)\psi(u) \big) \\ &= -\frac{3}{2}u - \left(9a + \frac{3}{2}\right)u^2 \\ &\quad - (9a + 12a^2 + 12b)u^3 + \mathcal{O}\big(u^4\big). \end{aligned} \quad (2.6)$$

By matching the u^2-terms of (2.5) and (2.6) we obtain $a = -\frac{1}{16}$. By then matching the u^3-terms we obtain $b = \frac{3}{64}$.

Exercise 2.1. For the system (2.1), similarly compute $W^u(1,0)$ to third order.

3 Towards a proof of the stable manifold theorem

- We first consider the affine non-autonomous system
$$\dot{\mathbf{x}} = A\mathbf{x} + \gamma(t), \tag{3.1}$$
where $\gamma : \mathbb{R} \to \mathbb{R}^n$. We suppose A has no eigenvalues with zero real part (so that $\mathbf{0}$ is a hyperbolic equilibrium for (3.1) when γ is the zero function). Then $E^c(\mathbf{0}) = \varnothing$, and every $\mathbf{x} \in \mathbb{R}^n$ can be uniquely written as $\mathbf{x} = \mathbf{x}_s + \mathbf{x}_u$ where $\mathbf{x}_s \in E^s(\mathbf{0})$ and $\mathbf{x}_u = E^u(\mathbf{0})$. We write
$$\mathbf{x}_s = \pi_s(\mathbf{x}), \qquad \mathbf{x}_u = \pi_u(\mathbf{x}), \tag{3.2}$$
where $\pi_s : \mathbb{R}^n \to E^s(\mathbf{0})$ and $\pi_u : \mathbb{R}^n \to E^u(\mathbf{0})$ are the appropriate projection operators.

Lemma 3.1. *Suppose A has no eigenvalues with zero real part and $\gamma : \mathbb{R} \to \mathbb{R}^n$ is continuous and bounded on $[0, \infty)$. Let $\sigma \in E^s(\mathbf{0})$. Then there exists unique $\chi \in E^u(\mathbf{0})$ such that, with $\mathbf{x}_0 = \sigma + \chi$, the solution to (3.1) with $\mathbf{x}(0) = \mathbf{x}_0$ is bounded. Moreover, this solution can be written as*
$$\mathbf{x}(t) = \mathrm{e}^{tA}\sigma + \int_0^t \mathrm{e}^{(t-s)A}\pi_s(\gamma(s))\,ds - \int_t^\infty \mathrm{e}^{(t-s)A}\pi_u(\gamma(s))\,ds. \tag{3.3}$$

Proof. The solution to (3.1) with an arbitrary initial condition $\mathbf{x}(\tau) = \mathbf{x}_\tau$ is
$$\mathbf{x}(t) = \mathrm{e}^{(t-\tau)A}\mathbf{x}_\tau + \int_\tau^t \mathrm{e}^{(t-s)A}\gamma(s)\,ds, \tag{3.4}$$
which can be seen by simply checking that it works, and obtained constructively by using an integrating factor.

We write $\mathbf{x}(t) = \pi_s(\mathbf{x}(t)) + \pi_u(\mathbf{x}(t))$, and consider these two projections separately.

By using (3.4) with $\tau = 0$, and noting $\pi_s(\mathbf{x}(0)) = \sigma$, we have
$$\pi_s(\mathbf{x}(t)) = \mathrm{e}^{tA}\sigma + \int_0^t \mathrm{e}^{(t-s)A}\pi_s(\gamma(s))\,ds. \tag{3.5}$$
We now show that (3.5) is bounded. Let $\alpha > 0$ be such that $|\mathrm{Re}(\lambda)| > \alpha$ for every eigenvalue λ of A. Since $\gamma(t)$ is bounded there exists $M \in \mathbb{R}$ such that $\|\gamma(t)\| \le M$ for all $t \ge 0$. Using also (7.2) of Part I we obtain
$$\left\| \int_0^t \mathrm{e}^{(t-s)A}\pi_s(\gamma(s))\,ds \right\| \le \frac{K_s M}{\alpha},$$
which verifies that (3.5) is bounded.

Secondly, by applying the projection π_u to (3.4) we obtain
$$\pi_u(\mathbf{x}(t)) = \mathrm{e}^{tA}\Big(\mathrm{e}^{-\tau A}\pi_u(\mathbf{x}_\tau) + \int_\tau^t \mathrm{e}^{-sA}\pi_u(\gamma(s))\,ds\Big). \tag{3.6}$$
If (3.6) is to be bounded for all $t \ge 0$, we require the term inside the parentheses to limit to zero as $t \to \infty$. That is
$$\mathrm{e}^{-\tau A}\pi_u(\mathbf{x}_\tau) + \int_\tau^\infty \mathrm{e}^{-sA}\pi_u(\gamma(s))\,ds = 0.$$
This holds for any τ, thus we can put $\tau = t$ and solve for $\pi_u(\mathbf{x}_t)$ to obtain
$$\pi_u(\mathbf{x}_t) = -\int_t^\infty \mathrm{e}^{(t-s)A}\pi_u(\gamma(s))\,ds. \tag{3.7}$$
Hence the solution is indeed given by (3.3), and putting $t = 0$ in (3.7) gives
$$\chi = -\int_0^\infty \mathrm{e}^{-sA}\pi_u(\gamma(s))\,ds. \tag{3.8}$$
Thus if (3.3) is to be bounded we must have χ given by (3.8). Then using (7.3) of Part II and (3.7) we obtain
$$\|\pi_u(\mathbf{x}_t)\| \le \frac{K_u M}{\alpha},$$
which shows that (3.3) is bounded. □

- Next we outline the proof of Theorem 2.2 given in [1], pages 176–180. As with the proof of the Picard-Lindelöf theorem, the essential tool is the contraction mapping theorem. Note that we do not actually use Lemma 3.1; it only motivates the function T.

Outline of proof of Theorem 2.2. We can assume $\mathbf{x}^* = \mathbf{0}$ (in view of the coordinate change $\mathbf{x} \mapsto \mathbf{x} - \mathbf{x}^*$). Then $f(\mathbf{x}) = A\mathbf{x} + g(\mathbf{x})$ where $g(\mathbf{x}) = o(\mathbf{x})$.

By analogy with (3.3), for any $\sigma \in E^s(\mathbf{0})$ we define a function $T : C^0([0,\infty),\mathbb{R}^n) \to C^0([0,\infty),\mathbb{R}^n)$, by

$$T(u(t)) = \mathrm{e}^{tA}\sigma + \int_0^t \mathrm{e}^{(t-s)A}\pi_s(u(s))\,ds - \int_t^\infty \mathrm{e}^{(t-s)A}\pi_u(u(s))\,ds.$$

It can be shown that T is a forward invariant contraction on a set

$$V_\delta = \left\{ u \in C^0([0,\infty),\mathbb{R}^n) \;\middle|\; \|u\| \le \delta \right\},$$

where $\delta > 0$ is sufficiently small. The contraction mapping theorem is then used to show that T has a unique fixed point $u^*(t;\sigma)$.

We then see that $u^*(t;\sigma)$ is a solution to (1.1). Moreover, we can use a generalised form of Grönwall's inequality to show that $u^*(t;\sigma) \to \mathbf{0}$ as $t \to \infty$, hence $u^*(t;\sigma) \subset W^s(\mathbf{0})$. This shows that $W^s_{\rm loc}(\mathbf{0})$ is a graph over E^s.

Finally, the smoothness of $W^s_{\rm loc}(\mathbf{0})$ and its tangency to $E^s(\mathbf{0})$ can be shown by using a result that says that if a contraction map varies smoothly with respect to parameter values, then its fixed point also varies smoothly. □

4 Centre manifolds

- Now we consider non-hyperbolic equilibria which are more difficult to deal with because the local dynamics does not split so simply into stable and unstable components.

Theorem 4.1 (Centre manifold theorem)**.** *Let $\mathbf{x}^*$ be an equilibrium of* (1.1) *and suppose f is C^k ($k \ge 1$) in a neighbourhood of $\mathbf{x}^*$. Then there exists $\delta > 0$ such that within $B_\delta(\mathbf{x}^*)$ there exists*

i) a unique C^k invariant manifold tangent to $E^s(\mathbf{x}^*)$ and contained within $W^s(\mathbf{x}^*)$,

ii) a unique C^k invariant manifold tangent to $E^u(\mathbf{x}^*)$ and contained within $W^u(\mathbf{x}^*)$,

iii) and a C^{k-1} invariant manifold tangent to $E^c(\mathbf{x}^*)$.

- Invariant manifolds tangent to $E^c(\mathbf{x}^*)$ are called *centre manifolds.* Part (iii) of Theorem 4.1 tells us that centre manifolds exist. For a proof of Theorem 4.1, see, for instance, [2].

- Usually we cannot hope to compute centre manifolds exactly, we can only compute their Taylor expansions up to some order. Centre manifolds are usually not unique, but if f is C^∞ in a neighbourhood of $\mathbf{x}^*$ then all centre manifolds have the same Taylor expansion. For this reason we often (imprecisely) refer to 'the' centre manifold of an equilibrium.

- Next we provide an analogy of the Hartman-Grobman theorem for non-hyperbolic equilibria. To state the result we first make some assumptions justified via coordinate transformations.

- We assume f is C^1 and that $\mathbf{0}$ is an equilibrium of (1.1). We assume that at $\mathbf{0}$ the Jacobian takes the block diagonal form

$$Df(\mathbf{0}) = \begin{bmatrix} C & & \\ & S & \\ & & U \end{bmatrix}, \tag{4.1}$$

where all eigenvalues of C have zero real part, all eigenvalues of S have negative real part, and all eigenvalues of U have positive real part. Write $\mathbf{x} = (\mathbf{u},\mathbf{v},\mathbf{w})$, where the dimensions of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ match those of C, S, and U, respectively. Then (1.1) can be written as

$$\begin{aligned} \dot{\mathbf{u}} &= C\mathbf{u} + F(\mathbf{u},\mathbf{v},\mathbf{w}), \\ \dot{\mathbf{v}} &= S\mathbf{v} + G(\mathbf{u},\mathbf{v},\mathbf{w}), \\ \dot{\mathbf{w}} &= U\mathbf{w} + H(\mathbf{u},\mathbf{v},\mathbf{w}), \end{aligned} \tag{4.2}$$

where F, G, and H are $o(\mathbf{u},\mathbf{v},\mathbf{w})$.

By Theorem 4.1, any centre manifold of $\mathbf{0}$ can be written locally as

$$W^c(\mathbf{0}) = \left\{ \begin{bmatrix} \mathbf{u} \\ g(\mathbf{u}) \\ h(\mathbf{u}) \end{bmatrix} \;\middle|\; \mathbf{u} \in B_\delta(\mathbf{0}) \right\}, \tag{4.3}$$

for some $o(\mathbf{u})$ functions g and h and some $\delta > 0$.

Theorem 4.2 (Non-hyperbolic Hartman-Grobman)**.** *There exists a neighbourhood of* $\mathbf{0}$ *within which* (1.1) *is conjugate to*

$$\begin{aligned} \dot{\mathbf{u}} &= C\mathbf{u} + F(\mathbf{u},g(\mathbf{u}),h(\mathbf{u})), \\ \dot{\mathbf{v}} &= S\mathbf{v}, \\ \dot{\mathbf{w}} &= U\mathbf{w}. \end{aligned}$$

Example 4.1. Consider the system

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = f(x,y) = \begin{bmatrix}x^5 - y\\ x^5 + y\end{bmatrix}. \tag{4.4}$$

i) Write $W^c(0,0)$ as $y = \phi(x)$ and compute the first nonzero term in a Taylor series expansion of ϕ about $x = 0$.

ii) Derive an equation for the restriction of (4.4) to $W^c(0,0)$.

iii) Use your result from (ii) to describe $W^s(0,0)$ and $W^u(0,0)$.

i) We have $Df(0,0) = \begin{bmatrix}0 & -1\\ 0 & 1\end{bmatrix}$. The eigenvalues are

$$\lambda_1 = 0, \qquad \lambda_2 = 1,$$

and the corresponding eigenvectors are

$$v_1 = \begin{bmatrix}1\\ 0\end{bmatrix}, \qquad v_2 = \begin{bmatrix}1\\ -1\end{bmatrix}.$$

The centre manifold $W^c(0,0)$ is tangent to v_1 at $(0,0)$, hence has slope 0 here. Thus we can write

$$\phi(x) = ax^2 + bx^3 + cx^4 + dx^5 + \mathcal{O}(6), \tag{4.5}$$

where we have chosen to include coefficients up to order five due to a little foresight.

Since centre manifolds are invariants, $y(t) = \phi(x(t))$ for all t. Differentiation with respect to t gives

$$\dot{y} = \phi'(x)\dot{x}. \tag{4.6}$$

The LHS of (4.6) is

$$ax^2 + bx^3 + cx^4 + (d+1)x^5 + \mathcal{O}(6),$$

whilst the RHS is

$$-2a^2x^3 - 5abx^4 - (6ac + 3b^2)x^5 + \mathcal{O}(6).$$

Matching the x^2 to x^5 terms gives $a = 0$, $b = 0$, $c = 0$, and $d = -1$. Hence $\phi(x) = -x^5 + \mathcal{O}(6)$.

ii) On $W^c(0,0)$, we have

$$\dot{x} = x^5 - \phi(x) = 2x^5 + \mathcal{O}(6). \tag{4.7}$$

iii) Since $\lambda_2 > 0$, in the v_2 direction nearby orbits head away from $(0,0)$. From (4.7) we deduce that in the v_1 direction nearby orbits also head away from $(0,0)$. Therefore $W^s(0,0) = \varnothing$ and $W^u(0,0)$ includes a neighbourhood of $(0,0)$, minus the point $(0,0)$ itself.

References

[1] J.D. Meiss. *Differential Dynamical Systems.* SIAM, Philadelphia, 2007.

[2] C. Chicone. *Ordinary Differential Equations with Applications.* Springer, New York, 1999.