Notes for 160.734
Part V: Bifurcations

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Here we begin to look at bifurcations. These are critical parameter values at which the dynamics of a system changes in a fundamental way.

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1 Structural stability

- Consider
$$\dot{\mathbf{x}} = f(\mathbf{x}; \mu), \tag{1.1}$$
where $f : \mathcal{X} \times \mathbb{R}^m \to \mathcal{X}$. From an applied viewpoint, we wish to understand how the dynamics of (1.1) changes as we vary the vector-valued parameter μ in a continuous fashion.
- Often the dynamics of (1.1) remains essentially unchanged as we vary μ. That is, while equilibria move a bit and their eigenvalues change a bit, the overall dynamics is qualitatively the same. Fundamental changes in the dynamics typically occurs only at a discrete set of values of μ.

Definition 1.1. We say that (1.1) is *structurally stable* for a given value of μ, if there exists $\delta > 0$ such that $\dot{\mathbf{x}} = f(\mathbf{x}; \tilde{\mu})$ is topologically equivalent to $\dot{\mathbf{x}} = f(\mathbf{x}; \mu)$ for all $\tilde{\mu} \in B_\delta(\mu)$.

Definition 1.2. If (1.1) is not structurally stable for a given value of μ, then μ is said to be a *bifurcation* value.

Exercise 1.1. Consider the linear system
$$\dot{\mathbf{x}} = \begin{bmatrix} \mu & 2 \\ 2\mu & 3 \end{bmatrix} \mathbf{x}, \tag{1.2}$$
where $\mu \in \mathbb{R}$.

i) Determine the range of values of μ for which $\mathbf{0}$ is a (a) stable node, (b) stable focus, (c) unstable focus, (d) unstable node, (e) saddle.

ii) Use your answer to (i) to determine the bifurcation values of (1.2). Hint: use Theorem 4.2 from Part III.

- Roughly speaking, a bifurcation occurs when an invariant set undergoes a fundamental change. From now until §5, we are now going to look solely at bifurcations for which this invariant set is an equilibrium.
- The next result tells us that hyperbolic equilibria are structurally stable features of (1.1).

Theorem 1.1. *Suppose f is a C^k ($k \ge 1$) function of $\mathbf{x}$ and μ, and suppose $\mathbf{x}^*$ is a hyperbolic equilibrium of (1.1) for a given value of μ. Then there exists $\delta > 0$ and a unique C^k function $\phi : B_\delta(\mu) \to \mathcal{X}$ with $\phi(\mu) = \mathbf{x}^*$, such that $\phi(\tilde{\mu})$ is a hyperbolic equilibrium of $\dot{\mathbf{x}} = f(\mathbf{x}; \tilde{\mu})$ for all $\tilde{\mu} \in B_\delta(\mu)$.*

Exercise 1.2. Show that Theorem 1.1 is a simple consequence of the Implicit Function Theorem:

Theorem 1.2 (Implicit Function Theorem)**.** *Suppose $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is C^k ($k \ge 1$). Suppose $f(\mathbf{x}^*; \mathbf{y}^*) = \mathbf{0}$ and $\det(D_\mathbf{x} f) \ne 0$ at $(\mathbf{x}^*; \mathbf{y}^*)$. Then there exists $\delta > 0$ and a unique C^k function $\phi : B_\delta(\mathbf{y}^*) \to \mathbb{R}^n$ with $\phi(\mathbf{y}^*) = \mathbf{x}^*$ such that $f(\phi(\mathbf{y}); \mathbf{y}) = \mathbf{0}$ for all $\mathbf{y} \in B_\delta(\mathbf{y}^*)$.*

- In view of Theorem 1.1, if an equilibrium of (1.1) undergoes a bifurcation it must have an associated eigenvalue with zero real part.
- The simplest cases are that $Df(\mathbf{x}^*)$ has (i) an eigenvalue 0, and (ii) purely imaginary eigenvalues $\pm i\omega$, and, in each case, no other eigenvalues with zero real part.
- The first case gives rise to a saddle-node bifurcation, §2. The second case gives rise to a Hopf bifurcation, §3.
- Indeed these are the only two *codimension-one* bifurcations that equilibria undergo, unless the system has some type of symmetry or

degeneracy, where a bifurcation is said to be codimension-k if it is determined by k independent scalar conditions on the parameter values.

2 Saddle-node bifurcations

- For simplicity we consider the one-dimensional system
$$\dot{x} = f(x;\mu), \tag{2.1}$$
where $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

Theorem 2.1. *Consider* (2.1) *where f is C^k ($k \geq 2$). Suppose*

i) $f(0;0) = 0$ *($x = 0$ is an equilibrium when $\mu = 0$),*

ii) $\frac{\partial f}{\partial x}(0;0) = 0$ *(the associated eigenvalue is zero when $\mu = 0$),*

iii) $\frac{\partial f}{\partial \mu}(0;0) \neq 0$ *(transversality condition),*

iv) $\frac{\partial^2 f}{\partial x^2}(0;0) \neq 0$ *(non-degeneracy condition).*

Then there exists $\delta > 0$ and a unique C^k function $\xi : [-\delta, \delta] \to \mathbb{R}$ with
$$\xi(0) = 0,$$
$$\xi'(0) = 0,$$
$$\xi''(0) = -\frac{\frac{\partial^2 f}{\partial x^2}(0;0)}{\frac{\partial f}{\partial \mu}(0;0)},$$
such that $f(x;\xi(x)) = 0$ for all $x \in [-\delta, \delta]$.

Proof. By the assumptions on f we can write
$$f(x;\mu) = a_1\mu + a_2x^2 + a_3\mu x + a_4\mu^2 + o(2), \tag{2.2}$$
where $a_1, a_2 \neq 0$. By the implicit function theorem (in Theorem 1.2 we use $\mathbf{x} = \mu$ and $\mathbf{y} = x$) there exists $\delta > 0$ and a unique C^k function $\xi : [-\delta, \delta] \to \mathbb{R}$ with $\xi(0) = 0$ such that $f(x;\xi(x)) = 0$ for all $x \in [-\delta, \delta]$. Write
$$\xi(x) = b_1x + b_2x^2 + o\big(x^2\big). \tag{2.3}$$
By substituting (2.3) into (2.2) we obtain
$$f(x;\xi(x)) = a_1b_1x + \big(a_1b_2 + a_2 + a_3b_1 + a_4b_1^2\big)x^2 + o\big(x^2\big). \tag{2.4}$$
We need $f(x;\xi(x)) = 0$, so from the x-term in (2.4) we obtain $b_1 = 0$, and from the x^2-term in (2.4) we obtain $b_2 = -\frac{a_2}{a_1}$. This completes the proof (in particular observe $\xi''(0) = 2b_2$, $\frac{\partial^2 f}{\partial x^2}(0;0) = 2a_2$, and $\frac{\partial f}{\partial \mu}(0;0) = a_1$). □

- Locally, if $\xi''(0) < 0$ then (2.1) has two equilibria for $\mu < 0$ and no equilibria for $\mu > 0$, while if $\xi''(0) > 0$ then (2.1) has no equilibria for $\mu < 0$ and two equilibria for $\mu > 0$.
- One of these equilibria will be stable, the other will be unstable.
- A *bifurcation diagram* is a plot indicating the invariant sets (e.g. equilibria) of the system with a parameter (e.g. μ) on the horizontal axis and a variable (e.g. x) on the vertical axis. Traditionally, stable and unstable sets are indicated with solid and dashed lines respectively (or blue and red lines, respectively, if colour is available).

Exercise 2.1. Consider
$$\dot{x} = \mu + x - \ln(1 + x). \tag{2.5}$$
Equilibria satisfy $0 = \mu + x - \ln(1 + x)$. Unfortunately we cannot explicitly solve this equation for x as a function of μ. However, we can solve it for μ: we have $\mu = \xi(x)$ where
$$\xi(x) = -x + \ln(1 + x). \tag{2.6}$$

i) Show that (2.5) satisfies all the conditions of Theorem 2.1.

ii) Verify that (2.6) satisfies all the conclusions to the theorem.

iii) Draw a bifurcation diagram of (2.6).

- Roughly speaking, the *normal form* of a bifurcation is the simplest representative system exhibiting the bifurcation. The normal form for a saddle-node bifurcation is
$$\dot{x} = \mu - x^2. \tag{2.7}$$
- Now consider
$$\dot{x} = \mu x - x^2, \tag{2.8}$$
$$\dot{x} = \mu x - x^3. \tag{2.9}$$

These are examples of systems that satisfy conditions (i) and (ii) of Theorem 2.1 (i.e. $x = 0$

is an equilibrium with eigenvalue 0 when $\mu = 0$) but do not satisfy the remaining conditions of the theorem. They are important examples of degenerate saddle-node bifurcations that are common in systems with symmetry or degeneracy.

Exercise 2.2. Sketch a bifurcation diagram of (2.8). This is the normal form of a *transcritical bifurcation*.

Exercise 2.3. Sketch a bifurcation diagram of (2.9). This is the normal form of a *pitchfork bifurcation* (your bifurcation diagram should look like a pitchfork).

3 Hopf bifurcations

Here we present the Hopf bifurcation theorem[1] as given in [1]. For a comprehensive review of Hopf bifurcations, see [2].

- Let us start with the normal form for a Hopf bifurcation:

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}\mu x - \omega y - \alpha x(x^2+y^2)\\ \omega x + \mu y - \alpha y(x^2+y^2)\end{bmatrix}, \qquad (3.1)$$

  where $\mu \in \mathbb{R}$ is the primary bifurcation parameter and $\omega, \alpha \in \mathbb{R}$ are additional parameters.

- Notice that $(x, y) = (0, 0)$ is an equilibrium of (3.1) with eigenvalues $\pm i\omega$ when $\mu = 0$.

Exercise 3.1. Show that in polar coordinates (3.1) is given by

$$\dot{r} = \mu r - \alpha r^3, \qquad \dot{\theta} = \omega. \qquad (3.2)$$

Use (3.2) to draw two bifurcation diagrams of (3.1), showing x as a function of μ. Assume $\omega > 0$ and α are fixed, with $\alpha < 0$ for one diagram and $\alpha > 0$ for the other diagram. Make sure to indicate the stability of the invariant sets in your diagrams.

- We now consider a two-dimensional system

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = f(x, y; \mu) = \begin{bmatrix}f_1(x, y; \mu)\\ f_2(x, y; \mu)\end{bmatrix}, \qquad (3.3)$$

  as this is the fewest number of dimensions in which Hopf bifurcations can occur.

Theorem 3.1. *Consider* (3.3) *where f is C^k ($k \geq 3$). Suppose*

i) $f_1(0, 0; 0) = f_2(0, 0; 0) = 0$ ($(x, y) = (0, 0)$ is an equilibrium when $\mu = 0$),

ii) $Df(0, 0; 0) = \begin{bmatrix}0 & -\omega\\ \omega & 0\end{bmatrix}$ for some $\omega \neq 0$ (the Jacobian matches that of the normal form (3.1)*),*

iii) $b = \frac{\partial^2 f_1}{\partial\mu\partial x} + \frac{\partial^2 f_2}{\partial\mu\partial y} \neq 0$ (transversality condition),

iv) $a \neq 0$ (non-degeneracy condition),

where

$$\begin{aligned} a = {} & \tfrac{1}{16}\left(\frac{\partial^3 f_1}{\partial x^3} + \frac{\partial^3 f_2}{\partial x^2\partial y} + \frac{\partial^3 f_1}{\partial x\partial y^2} + \frac{\partial^3 f_2}{\partial y^3}\right)\\ & + \tfrac{1}{16\omega}\left[\frac{\partial^2 f_1}{\partial x\partial y}\left(\frac{\partial^2 f_1}{\partial x^2} + \frac{\partial^2 f_1}{\partial y^2}\right) - \frac{\partial^2 f_1}{\partial x^2}\frac{\partial^2 f_2}{\partial x^2}\right.\\ & \left. - \frac{\partial^2 f_2}{\partial x\partial y}\left(\frac{\partial^2 f_2}{\partial x^2} + \frac{\partial^2 f_2}{\partial y^2}\right) + \frac{\partial^2 f_1}{\partial y^2}\frac{\partial^2 f_2}{\partial y^2}\right]. \end{aligned} \qquad (3.4)$$

Then a periodic orbit bifurcates into $\mu < 0$ if $ab > 0$ and into $\mu > 0$ if $ab < 0$. The amplitude of the periodic orbit is asymptotically proportional to $\sqrt{|\mu|}$, while the period limits to $\frac{2\pi}{|\omega|}$ as $\mu \to 0$.

- If the periodic orbit is stable [resp. unstable] we say the Hopf bifurcation is *supercritical* [resp. *subcritical*]. The stability of the periodic orbit is opposite to the stability of the equilibrium on the side of the bifurcation that it exists. The sign of b determines which side of $\mu = 0$ the equilibrium is stable, thus the criticality of the Hopf bifurcation is determined by the sign of a.

Exercise 3.2. Show that (3.1) satisfies the conditions of Theorem 3.1.

Exercise 3.3. Consider the van der Pol oscillator

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}y\\ -x + 2\mu y - x^2 y\end{bmatrix}. \qquad (3.5)$$

Show that (3.5) satisfies the conditions of Theorem 3.1 and determine the criticality of the bifurcation.

[1]First proved in $\mathbb{R}^n$ by Eberhard Hopf (1902–1983), sometimes called the Andronov-Hopf bifurcation as it first proved in $\mathbb{R}^2$ by Aleksandr Andronov (1901–1952), and sometimes called the Poincaré-Andronov-Hopf bifurcation as Poincaré was aware of the result.

4 Extended centre manifolds and dimension reduction

- Suppose $\mathbf{x}^*$ is a non-hyperbolic equilibrium of (1.1) with $\mu = \mu^*$. In the previous set of notes we encountered the non-hyperbolic Hartman-Grobman theorem which tells us that, with $\mu = \mu^*$, the dynamics of (1.1) is essentially determined by its behaviour on the centre manifold.
- In order to investigate the dynamics for values of μ near μ^*, we simply apply the non-hyperbolic Hartman-Grobman theorem to the *extended system*:

$$\begin{aligned} \dot{\mathbf{x}} &= f(\mathbf{x}; \mu), \\ \dot{\mu} &= \mathbf{0}, \end{aligned} \tag{4.1}$$

  by treating μ as a variable.
- In this way, bifurcations, such as the saddle-node bifurcation (described above in one dimension) and the Hopf bifurcation (described above in two dimensions), can be understood for (1.1) in any number of dimensions. This is the technique of dimension reduction via a centre manifold analysis.
- If $\mu = \mu^*$ corresponds to a codimension-one bifurcation, then the dimension of the centre manifold of (1.1) with $\mu = \mu^*$ is one, and for simplicity we can assume $\mu \in \mathbb{R}$. Then the centre manifold of (4.1), referred to as the *extended centre manifold*, is two-dimensional.

Example 4.1. Consider

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} (2+\mu)x + y + x^2 - y^3 \\ 2x + (1+3\mu)y - 2xy \end{bmatrix}. \tag{4.2}$$

Notice that $(x, y) = (0, 0)$ is an equilibrium when $\mu = 0$. Here we will compute the extended centre manifold and use it determine the local dynamics.

The extended system is

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\mu} \end{bmatrix} = F(x, y, \mu) = \begin{bmatrix} (2+\mu)x + y + x^2 - y^3 \\ 2x + (1+3\mu)y - 2xy \\ 0 \end{bmatrix}.$$

The Jacobian of the extended system evaluated at $(x, y, \mu) = (0, 0, 0)$ is

$$DF(0,0,0) = \begin{bmatrix} 2 & 1 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{4.3}$$

The eigenvalues and eigenvectors of (4.3) are

$$\lambda_1 = 3, \qquad v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},$$

$$\lambda_2 = 0, \qquad v_2 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \qquad v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The centre subspace $E^c(0, 0, 0)$ is the span of v_2 and v_3 and can be written as $y = -2x$. Therefore $W^c(0, 0, 0)$ can be written as $y = \phi(x, \mu)$ where

$$\phi(x, \mu) = -2x + ax^2 + b\mu x + c\mu^2 + \mathcal{O}(3), \tag{4.4}$$

for some $a, b, c \in \mathbb{R}$.

We now write $\dot{y}$ in two different ways, and match terms, in order to evaluate a, b, and c. First, by substituting (4.4) for y in the right hand side of (4.2), after simplification we obtain

$$\dot{y} = (4 + a)x^2 + (-6 + b)\mu x + c\mu^2 + \mathcal{O}(3). \tag{4.5}$$

Second, by the chain rule, $\dot{y} = \frac{\partial \phi}{\partial x}\dot{x} + \frac{\partial \phi}{\partial \mu}\dot{\mu}$. Since $\dot{\mu} = 0$ we have

$$\begin{aligned} \dot{y} &= \frac{\partial \phi}{\partial x}\dot{x} \\ &= (-2 - 2a)x^2 + (-2 - 2b)\mu x - 2c\mu^2 + \mathcal{O}(3). \end{aligned} \tag{4.6}$$

Upon matching (4.5) and (4.6) we obtain

$$a = -2, \qquad b = \frac{4}{3}, \qquad c = 0. \tag{4.7}$$

We have thus computed $W^c(0, 0, 0)$ to second order.

By substituting $y = \phi(x, \mu)$ with (4.7) into (4.2), we see that on $W^c(0, 0, 0)$ we have

$$\dot{x} = -x^2 + \frac{7}{3}\mu x + \mathcal{O}(3). \tag{4.8}$$

It is then a simple exercise to compute the equilibria of (4.8) and draw a bifurcation diagram from which we can infer that $\mu = 0$ is a transcritical bifurcation.

Exercise 4.1. Here you will study the dynamics of a model of a semiconductor laser with feedback (given in [3]):

$$\begin{aligned}\dot{x} &= x(y-1), \\ \dot{y} &= \gamma\left(\delta_0 - y - xy + \frac{\alpha(x+z)}{1+s(x+z)}\right), \\ \dot{z} &= -\varepsilon(x+z).\end{aligned} \tag{4.9}$$

These equations are simplified and dimensionless; x represents photon density, y represents electron density, and z is a linear combination of current and x. To answer the following questions you may assume

$$\gamma = 0.001, \quad \delta_0 = 1.017, \quad \alpha = 1, \quad s = 11. \tag{4.10}$$

i) Suppose $\varepsilon = 0$ (with which the system is effectively two-dimensional). Calculate equilibria and their stability.

ii) Draw a bifurcation diagram of the system with $\varepsilon = 0$. What two bifurcations do you see in the diagram?

iii) Use MATLAB to explore the dynamics of the system with small $\varepsilon > 0$. Use the bifurcation diagram with $\varepsilon = 0$ to help explain your findings.

5 Global bifurcations

- Thus far we have only considered bifurcations that occur when an equilibrium loses hyperbolicity. These are examples of *local* bifurcations because, under generic conditions, only the dynamics in a neighbourhood of the equilibrium undergoes a topological change.
- In contrast a *global* bifurcation affects the dynamics, well, globally. Typically these can only be identified numerically.
- An important example of a global bifurcation is a *homoclinic bifurcation*. A homoclinic bifurcation occurs when a periodic orbit collides with an equilibrium and turns into a homoclinic orbit. When this happens under generic conditions, the stable and unstable manifolds of the equilibrium suddenly access different parts of phase space.

Exercise 5.1. Consider the forced van der Pol system

$$\begin{bmatrix}\dot{x} \\ \dot{y}\end{bmatrix} = \begin{bmatrix}\frac{1}{\varepsilon}\left(y - x^3 - a_1 x - a_0\right) \\ b_2 x^2 + b_1 x - y\end{bmatrix}, \tag{5.1}$$

studied in [4]. Here we fix

$$\varepsilon = 0.5, \quad a_1 = -1, \quad b_1 = -4, \quad b_2 = 5, \tag{5.2}$$

and use $a_0 \in \mathbb{R}$ as a bifurcation parameter.

i) Plot a basic bifurcation diagram showing equilibria and identify the a_0 values of two saddle-node bifurcations.

ii) Use MATLAB to numerically identify a supercritical Hopf bifurcation and a homoclinic bifurcation. Add these features to your bifurcation diagram.

6 Codimension-two bifurcations

- Thus far we have only considered codimension-one bifurcations. A codimension-two bifurcation involves two scalar conditions. Examples include a *Takens-Bogdanov bifurcation* (where an equilibrium has an associated eigenvalue of 0 with algebraic multiplicity two) and a *Bautin bifurcation* (what would be a Hopf bifurcation except that the criticality parameter (3.4) is zero).
- To summarise the dynamics near a codimension-two bifurcation, one usually plots a two-parameter bifurcation diagram (sometimes called a bifurcation set). This uses a parameter on each axis. Thus curves in two-parameter bifurcation diagrams represent codimension-one bifurcations.

Exercise 6.1. Consider

$$\dot{x} = f(x; \mu_1, \mu_2) = \mu_1 + \mu_2 x - x^3. \tag{6.1}$$

With $(\mu_1, \mu_2) = (0, 0)$, $x = 0$ is an equilibrium with an associated eigenvalue of 0. This would be a saddle-node bifurcation except that the non-degeneracy condition is not satisfied because $\frac{\partial^2 f}{\partial x^2}(0; 0, 0) = 0$. Equation (6.1) is the normal form for a *cusp bifurcation*.

i) Show that (6.1) has saddle-node bifurcations on the curve $27\mu_1^2 - 4\mu_2^3 = 0$.

ii) Draw a two-parameter bifurcation diagram: show the curve $27\mu_1^2 - 4\mu_2^3 = 0$ and overlay a few small representative phase portraits.

References

[1] P. Glendinning. *Stability, Instability and Chaos: An Introduction to the Theory of Nonlinear Differential Equations.* Cambridge University Press, New York, 1999.

[2] J.E. Marsden and M. McCracken. *The Hopf Bifurcation and its Applications*. Springer-Verlag, New York, 1976.

[3] K. Al-Naimee, F. Marino, M. Ciszak, R. Meucci, and F.T. Arecchi. Chaotic spiking and incomplete homoclinic scenarios in semiconductor lasers with optoelectronic feedback. *New Journal of Physics*, 11(7):073022, 2009.

[4] M.T.M. Koper. Some simple bifurcation sets of an extended Van der Pol model and their relation to chemical oscillators. *J. Chem. Phys.*, 102:5278–5287, 1995.