

Notes for 160.734

Part VI: A First Look at Chaos

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Chaos theory is one the great triumphs of 20th century science. Chaos was first recognised by Poincaré in the 1880's while attempting to solve the three-body problem. Many notable advances were then made in the 1920's and 1930's in the context of ergodic theory (which we will come to later). However, it wasn't until computer simulations became available that the true nature, importance, and ubiquity of chaos was realised. In 1963 Lorenz discovered how chaos can make long-term weather prediction impossible [1]. The term 'chaos' was coined by Yorke in 1975 and first given a mathematical definition in [2].

Here we explore chaos in ODEs. Today there is no universally accepted definition of chaos, and indeed to quote from a recent article by Sander and Yorke [3]: *the concept of chaos is too big for one single mathematical definition to suffice*. For a popular science introduction to chaos refer to the marvelous book of Gleick [4].

1 Sensitive dependence and transitivity

- As before we consider

$$\dot{\mathbf{x}} = f(\mathbf{x}), \quad (1.1)$$

where $f : \mathcal{X} \rightarrow \mathcal{X}$ and let $\varphi_t(\mathbf{x})$ denote the solution.

Definition 1.1. Let $\Omega \subset \mathcal{X}$ be an invariant set of (1.1). Then (1.1) exhibits *sensitive dependence on initial conditions* on Ω if there exists $r > 0$ such that for all $\mathbf{x} \in \Omega$ and all $\varepsilon > 0$ there exists $\mathbf{y} \in B_\varepsilon(\mathbf{x}) \cap \Omega$ such that $|\varphi_t(\mathbf{x}) - \varphi_t(\mathbf{y})| > r$ for some $t \geq 0$.

Example 1.1. Here we show that

$$\dot{x} = 3x, \quad (1.2)$$

exhibits sensitive dependence on initial conditions.

The solution to (1.2) is $\varphi_t(x) = xe^{3t}$. Thus $\varphi_t(x + \varepsilon) = (x + \varepsilon)e^{3t} = \varphi_t(x) + \varepsilon e^{3t}$. Thus, for any $\varepsilon > 0$, we can make $|\varphi_t(x) - \varphi_t(x + \varepsilon)| = \varepsilon e^{3t}$ as large as we like by choosing t sufficiently large. Thus Definition 1.1 can be satisfied with, say, $r = 1$.

Definition 1.2. Let $\Omega \subset \mathcal{X}$ be an invariant set of (1.1). Then (1.1) is *transitive* on Ω if for all nonempty open $U, V \subset \Omega$, there exists $t > 0$ such that $\varphi_t(U) \cap V \neq \emptyset$.

- Note, U and V are open with respect to the induced metric on Ω . Specifically a set $U \subset \Omega$ is *open* if for every $x \in U$ there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \cap \Omega \subset U$.

Exercise 1.1. Consider

$$\begin{aligned} \dot{\theta}_1 &= 1, \\ \dot{\theta}_2 &= \nu, \end{aligned} \quad (1.3)$$

where $\theta_1, \theta_2 \in [0, 2\pi)$ (thus phase space is a torus). Show (1.3) is transitive if and only if the parameter ν is not a rational multiple of π .

Lemma 1.1. Let $\Omega \subset \mathcal{X}$ be an invariant set of (1.1) where the manifold $\mathcal{X}$ is second countable¹. If (1.1) is transitive on Ω then it has a dense forward orbit in Ω .

Proof. Since $\mathcal{X}$ is second countable it is *separable* which means there exists a countable collection of points $\{y_n\}_{n=1}^\infty$ that are dense in Ω . For each n , let $U_1 = B_{\frac{1}{n}}(y_n) \cap \Omega$. By transitivity there exists $t_2 > 0$ such that $\varphi_{t_2}(U_1) \cap U_2 \neq \emptyset$. Then there exists open $V_2 \subset U_1$, with diameter at most $\frac{1}{2}$, such that $\varphi_{t_2}(V_2) \subset U_2$. Similarly there exists $t_3 > 0$ such that $\varphi_{t_3}(V_2) \cap U_3 \neq \emptyset$. Then there

¹A topological space $\mathcal{X}$ is *second countable* if there exists a countable collection of open subsets of $\mathcal{X}$ such that any open subset of $\mathcal{X}$ can be written as a union of elements of this collection. For example $\mathbb{R}^n$ (with the usual topology) is second countable for any $n \geq 1$.

exists open $V_3 \subset V_2$, with diameter at most $\frac{1}{3}$, such that $\varphi_{t_3}(V_3) \subset U_3$. Continuing we obtain times t_n and nested open sets V_n , with diameter at most $\frac{1}{n}$, such that $\varphi_{t_m}(V_n) \subset U_m$, for all $m = 2, \dots, n$. Write $\{\mathbf{x}\} = \bigcap_{n=2}^{\infty} V_n$. Then, by construction, $\Gamma_{\mathbf{x}}^+$ is dense in Ω . $\square$

- We are now ready to provide a definition of chaos following [5].

Definition 1.3. The system (1.1) is *chaotic* on a compact invariant set Ω if it is transitive and exhibits sensitive dependence on initial conditions on Ω .

Exercise 1.2. Consider the Lorenz system [1]

$$\begin{aligned}\dot{x} &= \sigma(y - x), \\ \dot{y} &= rx - xz - y, \\ \dot{z} &= xy - bz.\end{aligned}\tag{1.4}$$

- Treat $\sigma > 0$ and $b > 0$ as fixed and $r > 0$ as a bifurcation parameter. Find all equilibria of (1.4). What type of bifurcation is $r = 1$?
- Use MATLAB to compute orbits of (1.4) with $\sigma = 10$, $b = \frac{8}{3}$ and several values of $r \in [1, 28]$. Observe two Hopf bifurcations (occurring at the same value of r) and convince yourself that (1.4) is chaotic, according to Definition 1.3, with $r = 28$. Chaos with $r = 28$ was formally proved by Tucker in 2002, see [6].

2 Fractals and dimension

- An attractor is said to be *strange* if it is a ‘fractal’.
- Fractals are difficult to define precisely. Essentially they involve a pattern that repeats on different scales.
- Here we provide a simple notion of dimension which is useful for describing the size of a fractal. The Hausdorff dimension is a similar but more sophisticated notion of dimension.

Definition 2.1. Let $\Omega \subset \mathbb{R}^n$. For each $\varepsilon > 0$ let $N(\varepsilon)$ be the minimum number of n -dimensional cubes of edge length ε needed to cover Ω . Then

²Two maps $g_1 : \mathcal{Y}_1 \rightarrow \mathcal{Y}_1$ and $g_2 : \mathcal{Y}_2 \rightarrow \mathcal{Y}_2$ are said to be *conjugate* if there exists a homeomorphism $h : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$ such that $h(g_1(\mathbf{y})) = g_2(h(\mathbf{y}))$ for all $\mathbf{y} \in \mathcal{Y}_1$.

the *box dimension* (or *Minkowski-Bouligand dimension*) of Ω is

$$d_{\text{box}} = \lim_{\varepsilon \rightarrow 0} \frac{\ln(N(\varepsilon))}{\ln(\frac{1}{\varepsilon})}.\tag{2.1}$$

Exercise 2.1. Let $S_0 \subset \mathbb{R}^2$ be an equilateral triangle with edge length 1. For each $n = 1, 2, \dots$ construct a set $S_n \subset \mathbb{R}^2$ by replacing the middle third of each side of S_{n-1} with an equilateral triangle on the outside of S_{n-1} . Determine the box dimension of $\lim_{n \rightarrow \infty} S_n$ (the Koch snowflake).

3 Poincaré maps

- A discrete-time dynamical system

$$\mathbf{x}_{i+1} = g(\mathbf{x}_i),\tag{3.1}$$

where $g : \mathcal{Y} \rightarrow \mathcal{Y}$ for some manifold $\mathcal{Y}$, is referred to as a *map*.

- Maps arise from the system (1.1) as the first return of orbits to a cross-section of $\mathcal{X}$. Such maps are called Poincaré maps.

Definition 3.1. Let $\Pi \subset \mathcal{X}$ be smooth, connected, and $(n-1)$ -dimensional, and suppose $f(\mathbf{x})$ is not tangent to Π at $\mathbf{x}$ for all $\mathbf{x} \in \Pi$ (such a set Π is called a *Poincaré section*). We define the *Poincaré map* $P : \Pi \rightarrow \Pi$ by letting $P(\mathbf{x}) = \varphi_t(\mathbf{x})$, for the smallest $t > 0$ for which $\varphi_t(\mathbf{x}) \in \Pi$, if such a t exists, for all $\mathbf{x} \in \Pi$.

- Poincaré maps are usually only well-defined locally because if we follow the cross-section Π far enough we often come to a point $\mathbf{x} \in \Pi$ where $f(\mathbf{x})$ is tangent to Π or the forward orbit of $\mathbf{x}$ does not return to Π .
- The motivation here is that by analysing a Poincaré map of (1.1) we can hope to better understand oscillatory dynamics. For example, if (1.1) has a periodic orbit that intersects Π at a single point $\mathbf{y}$, then $\mathbf{y}$ is a fixed point of P (i.e. $P(\mathbf{y}) = \mathbf{y}$). As we will see, by doing a basic stability analysis of $\mathbf{y}$ for the map P (by computing the eigenvalues of $DP(\mathbf{y})$) we can determine the stability of the periodic orbit for the system (1.1).

- The choice of Poincaré section is not important in the sense that, under certain restrictions, any two Poincaré maps are *conjugate*².
- As an example we consider

$$\begin{aligned}\dot{x} &= -y - z, \\ \dot{y} &= x + ay, \\ \dot{z} &= b + z(x - c),\end{aligned}\tag{3.2}$$

where $a, b, c \in \mathbb{R}$ are parameters. This is known as the *Rössler system* as it was introduced by Otto Rössler [7] who had the aim of studying a chaotic system simpler than the Lorenz system (indeed (3.2) contains only a single nonlinear term).

- Fig. 1 shows a typical forward orbit of (3.2) (with transients removed) with

$$a = 0.2, \quad b = 0.2, \quad c = 5.7, \tag{3.3}$$

as used in [7]. The direction of motion is anti-clockwise.

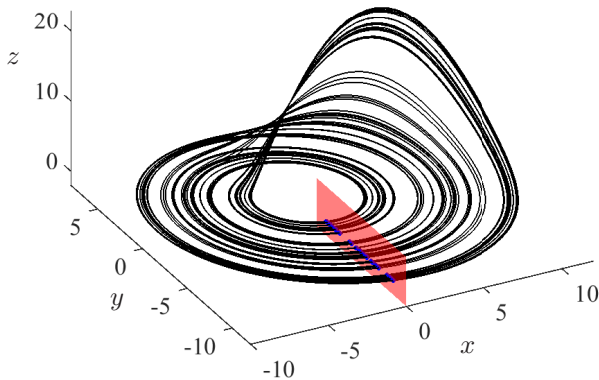


Figure 1: The Rössler attractor (more precisely part of a numerically computed orbit is shown).

- Fig. 1 essentially shows the attractor of (3.2) with (3.3). Roughly speaking, while near the $z = 0$ plane, orbits on the attractor spiral outwards until they are ejected upwards, twisted, and folded back down towards the $z = 0$ plane.
- We consider the Poincaré section defined by $x = 0$ and $y < 0$ as shown in Fig. 1. The orbit intersects this section at a sequence of points $(x, y, z) = (0, y_i, z_i)$.
- The corresponding Poincaré map is an invertible two-dimensional map. However, due to the strong contraction to the $z = 0$ plane, this map is well-approximated by a one-dimensional map, as indicated in Fig. 2. Notice that this one-dimensional map is non-invertible, and this is what allows the dynamics of the Rössler system to be complicated (chaotic in fact).

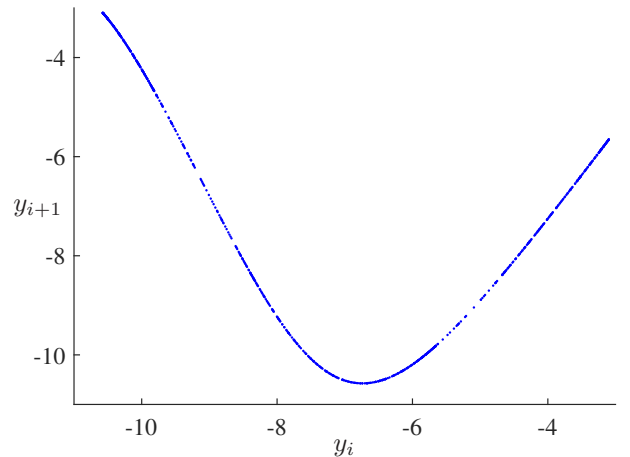


Figure 2: The y -values of intersections of the orbit shown in Fig. 1 with the Poincaré section $x = 0$, $y < 0$.

References

- [1] E.N. Lorenz. Deterministic nonperiodic flow. *J. Atmos. Sci.*, 20:130–141, 1963.
- [2] T. Li and J.A. Yorke. Period three implies chaos. *Amer. Math. Monthly.*, 82(10):985–992, 1975.
- [3] E. Sander and J.A. Yorke. Chaos and its manifestations. *SIAM News.*, 48(7), 2015.
- [4] J. Gleick. *Chaos: making a new science*. Viking, New York, 1987.
- [5] J.D. Meiss. *Differential Dynamical Systems*. SIAM, Philadelphia, 2007.
- [6] W. Tucker. A rigorous ODE solver and Smale’s 14th problem. *Found. Comput. Math.*, 2(1):53–117, 2002.
- [7] O.E. Rössler. An equation for continuous chaos. *Phys. Lett.*, 57A:397–398, 1976.