Notes for 160.734
Part VII: Maps

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Maps model discrete-time phenomena and arise as return maps (Poincaré maps) in ODE systems. As we saw in Part VI, chaotic dynamics in the Rössler system is well-approximated by a one-dimensional unimodal map. It is therefore important to study maps, and with one-dimensional unimodal maps we hope to obtain a deeper understanding of chaos.

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1 Dynamical systems definitions

- We consider a map

$$\mathbf{x} \mapsto f(\mathbf{x}), \tag{1.1}$$

where $f : \mathcal{X} \to \mathcal{X}$ is continuous. We write

$$f^n = \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}} \tag{1.2}$$

for the composition of f with itself n times. Given $\mathbf{x} \in \mathcal{X}$, the point $f^n(\mathbf{x})$ is called the $n^{\rm th}$ iterate of $\mathbf{x}$ under f.

- Many dynamical systems concepts that we have studied for ODEs generalise naturally to maps. The difference is that now time is discrete instead of continuous. Here we redefine some key definitions for (1.1).

Definition 1.1. A *fixed point* of (1.1) is a point $\mathbf{x}^* \in \mathcal{X}$ for which $f(\mathbf{x}^*) = \mathbf{x}^*$.

Definition 1.2. A *forward invariant region* of (1.1) is a set $\Omega \subset \mathcal{X}$ for which $f(\Omega) \subset \Omega$. A *trapping region* is a non-empty, compact set $\Omega \subset \mathcal{X}$ for which $f(\Omega) \subset {\rm int}(\Omega)$.

Definition 1.3. A set $\Lambda \subset \mathcal{X}$ is said to be an *attracting set* of (1.1) if there exists a trapping region Ω such that

$$\Lambda = \bigcap_{n=0}^{\infty} f^n(\Omega).$$

An *attractor* is an attracting set that contains a dense orbit.

2 Aspects of one-dimensional maps

- As an example, consider the map

$$f(x) = 1 - 6x^2 + 6x^3, \tag{2.1}$$

shown in Fig. 1.

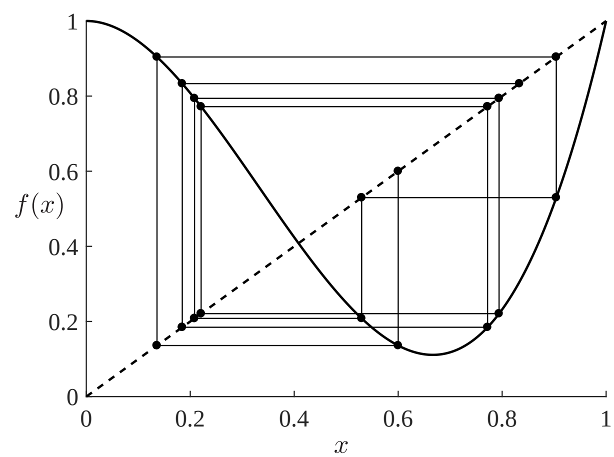

Figure 1: A cobweb diagram for the map (2.1).

- We can sketch orbits of a one-dimensional map, such as (2.1), as follows. We first draw the 45° line, $f(x) = x$. Given an initial value u, we start at the point (u, u), then draw a vertical line segment to the graph of f, then continue with a horizontal line segment back to the 45° line. These two line segments take us to the point $\big(f(u), f(u)\big)$. By repeating these steps, as shown in Fig. 1 using $u = 0.6$, we can compute the forward orbit of u. This is known as a *cobweb diagram*.

- Cobweb diagrams allow us to effectively simulate iterations of the map by hand and can be used to determine the qualitative nature of the dynamics without the need for a computer.

Exercise 2.1. For the map

$$f(x) = \begin{cases} \frac{1}{4}x + 1, & x \le 0, \\ -2x + 1, & x \ge 0, \end{cases} \tag{2.2}$$

draw a cobweb diagram to illustrate the forward orbit of $u = 0$. What do you think the orbit converges to?

- The next result (first given in [1]) involves the following alternate ordering of the positive integers:

$$\begin{aligned} &3 \rhd 5 \rhd 7 \rhd 9 \rhd 11 \rhd \cdots \\ &\rhd 2 \cdot 3 \rhd 2 \cdot 5 \rhd 2 \cdot 7 \rhd 2 \cdot 9 \rhd 2 \cdot 11 \rhd \cdots \\ &\rhd 2^2 \cdot 3 \rhd 2^2 \cdot 5 \rhd 2^2 \cdot 7 \rhd 2^2 \cdot 9 \rhd 2^2 \cdot 11 \rhd \cdots \\ &\cdots \\ &\rhd 32 \rhd 16 \rhd 8 \rhd 4 \rhd 2 \rhd 1 \end{aligned}$$

Theorem 2.1 (Sharkovskii's theorem)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Suppose f has a period-n orbit (i.e. $f^n(x) = x$, for some $x \in \mathbb{R}$). Then, for all $m \lhd n$, f also has a period-m orbit.*

- In particular, this theorem tells us that if f has a period-3 orbit, then it has orbits of every period! In [2], Li and Yorke went one step further and showed that in this instance f must also be chaotic.
- We will not prove Sharkovskii's theorem, just give a brief example to illustrate the type of arguments that are involved. More details are presented in [3].

Example 2.1. Suppose a continuous map $f : \mathbb{R} \to \mathbb{R}$ has a period-n solution, for some $n > 1$. Show that f also has a fixed point.

Let $x_{\rm min}$ and $x_{\rm max}$ denote the minimum and maximum points of the period-n solution. Then $f(x_{\rm min}) > x_{\rm min}$, whilst $f(x_{\rm max}) < x_{\rm max}$. Since f is continuous, by the intermediate value theorem $f(x) = x$ for some $x \in (x_{\rm min}, x_{\rm max})$.

- The next discussion concerns maps on $[0, 1]$. This is because we are commonly only interested in the dynamics of a map in some forward invariant region. For a one-dimensional map we can usually assume this region is an interval and by a change of coordinates can assume it is $[0, 1]$.

Definition 2.1. A continuous map $f : [0, 1] \to [0, 1]$ is said to be *unimodal* if there exists $c \in (0, 1)$ such that either $f(x) < c$ for all $x \ne c$ or $f(x) > c$ for all $x \ne c$. Furthermore, f is said to be *S-unimodal* if it is C^3 and the *Schwarzian derivative*

$$Sf = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2, \tag{2.3}$$

is non-positive in $[0, c) \cup (c, 1]$.

Exercise 2.2. Consider $f(x) = \frac{ax+b}{cx+d}$, where $a, b, c, d \in \mathbb{R}$. Show that $(Sf)(x) = 0$ for all $x \in \mathbb{R}$ with $cx + d \ne 0$.

Exercise 2.3. Argue that to understand S-unimodal maps it suffices to assume $f(c) = 1$ and $f(1) = 0$.

- The next result, taken from [4], essentially says that any S-unimodal map with non-degenerate[1] extremum has a unique attractor and that this attractor has one of three possible structures.

Theorem 2.2. *Let $f : [0, 1] \to [0, 1]$ be a S-unimodal map with extremum $c \in (0, 1)$. Suppose $f(c) = 1$, $f(1) = 0$ and $f''(c) \ne 0$. Then there exists a unique attractor $\Lambda \subset [0, 1]$ such that $\Lambda = \omega(x)$ for almost all[2] $x \in [0, 1]$ and either*

i) Λ is periodic solution,

ii) Λ is a cycle of disjoint intervals[3], or

iii) Λ is a Feigenbaum-like attractor[4].

[1] Here non-degenerate means $f''(c) \ne 0$. If $f''(c) = 0$ additional complications are possible, see [5].

[2] The phrase 'almost all $x \in [0, 1]$' will be defined in Part VIII. It essentially means 'with probability 1 for randomly chosen $x \in [0, 1]$'.

[3] That is, $\Lambda = \bigcup_{i=0}^{n-1} I_i$, where $I_0, \ldots, I_{n-1}$ are disjoint intervals in $[0, 1]$ and $f(I_0) = I_1, f(I_1) = I_2, \ldots, f(I_{n-1}) = I_0$.

[4] A Feigenbaum-like attractor is a certain type of Cantor set, see for instance [4].

3 Fixed points and stability

Definition 3.1. A fixed point $\mathbf{x}^*$ of (1.1) is said to be *Lyapunov stable* if for all $\varepsilon > 0$ there exists $\delta > 0$ such that $f^n(\mathbf{x}) \in B_\varepsilon(\mathbf{x}^*)$ for all $\mathbf{x} \in B_\delta(\mathbf{x}^*)$ and all $n \geq 0$.

Definition 3.2. A fixed point $\mathbf{x}^*$ of (1.1) is said to be *asymptotically stable* if it is Lyapunov stable and there exists $\delta > 0$ such that $f^n(\mathbf{x}) \to \mathbf{x}^*$ as $n \to \infty$ for all $\mathbf{x} \in B_\delta(\mathbf{x}^*)$.

Exercise 3.1. Consider the linear map

$$\mathbf{x} \mapsto A\mathbf{x}, \tag{3.1}$$

where A is a real-valued $N \times N$ matrix. Notice that for any $\mathbf{x} \in \mathbb{R}^N$ we have $f^n(\mathbf{x}) = A^n\mathbf{x}$.

i) Show that if $|\lambda| < 1$ for every eigenvalue λ of A, then $\mathbf{0}$ is an asymptotically stable fixed point of (3.1).

ii) Show that if $|\lambda| > 1$ for some eigenvalue λ of A, then $\mathbf{0}$ is not a Lyapunov stable fixed point of (3.1).

- If f is smooth, or at least C^1, then, analogous to the Hartman-Grobman theorem for ODEs, the dynamics near a fixed point of (1.1) are well-approximated by the linearisation of (1.1) about the fixed point. For instance, we have the following result which generalises the previous exercise.

Theorem 3.1. *Let $\mathbf{x}^*$ be a fixed point of (1.1) and suppose f is C^1.*

i) If $|\lambda| < 1$ for every eigenvalue λ of $Df(\mathbf{x}^*)$, then $\mathbf{x}^*$ is an asymptotically stable fixed point of (1.1).

ii) If $|\lambda| > 1$ for some eigenvalue λ of $Df(\mathbf{x}^*)$, then $\mathbf{x}^*$ is not a Lyapunov stable fixed point of (1.1).

- The eigenvalues of $Df(\mathbf{x}^*)$ are called the *stability multipliers* of $\mathbf{x}^*$.
- Here we consider the linear map (3.1) in two-dimensions and write

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \tag{3.2}$$

The stability multipliers of the fixed point $\mathbf{0}$ are the roots of $\lambda^2 - \tau\lambda + \delta = 0$ where

$$\tau = a + d, \qquad \delta = ad - bc. \tag{3.3}$$

- Solving $|\lambda| = 1$ produces three cases:

i) $\lambda = 1$; here $\delta = \tau - 1$.

ii) $\lambda = -1$; here $\delta = -\tau - 1$.

iii) $\lambda = \mathrm{e}^{\mathrm{i}\phi}$ for some $\phi \in (0, \pi)$; here $\delta = 1$ and $\tau \in (-2, 2)$.

The origin $\mathbf{0}$ is stable in the triangle of the (τ, δ)-plane bounded by the lines (i), (ii), and (iii).

4 Structural stability

- To motivate the ideas in this section, consider the one-dimensional linear map

$$f(x) = \tfrac{x}{2}. \tag{4.1}$$

Here 0 is the unique fixed point of (1.1) and is asymptotically stable (the stability multiplier is $\frac{1}{2}$).

- Intuitively we think of the stable fixed point 0 as a 'robust' feature of (4.1) in the sense that if we change the map by a small amount, then a single stable fixed point should persist. But we must be careful, as the following example shows. For any $\varepsilon > 0$, let $k_\varepsilon = \frac{1}{2\mathrm{e}\varepsilon^2}$. The range of the function

$$g_\varepsilon(x) = x\mathrm{e}^{-k_\varepsilon x^2}, \tag{4.2}$$

is $[-\varepsilon, \varepsilon]$, that is $\|g_\varepsilon\|_\infty = \varepsilon$. Yet the map

$$f(x) = \tfrac{x}{2} + g_\varepsilon(x), \tag{4.3}$$

has three fixed points including 0 which is now unstable (the stability multiplier is $\frac{3}{2}$). In summary, we have changed the qualitative behaviour of the map near an asymptotically stable fixed point by applying an arbitrarily small continuous perturbation.

- The catch in this example is that the derivative of the perturbation is not small (specifically $\frac{dg_\varepsilon}{dx}(0) = 1$). This highlights the fact that the class of perturbations that we allow is critically important to the robustness and structural stability of the features of a map.

- Structural stability was introduced for families of ODEs in Part V. Here we reformulate this for (i) families of maps, and (ii) perturbations of a single map.

Definition 4.1. Two maps $f_1 : \mathcal{X}_1 \to \mathcal{X}_1$ and $f_2 : \mathcal{X}_2 \to \mathcal{X}_2$ are said to be *conjugate* if there exists a homeomorphism $h : \mathcal{X}_1 \to \mathcal{X}_2$ such that

$$h(f_1(\mathbf{y})) = f_2(h(\mathbf{y})), \quad \text{for all } \mathbf{y} \in \mathcal{X}_1 \,. \tag{4.4}$$

Definition 4.2. A family of maps

$$\mathbf{x} \mapsto f(\mathbf{x}; \mu), \tag{4.5}$$

where $f : \mathcal{X} \times \mathbb{R}^m \to \mathcal{X}$, is *structurally stable* at a given value of μ if $\mathbf{x}_{i+1} = f(\mathbf{x}_i; \tilde{\mu})$ is conjugate to $\mathbf{x}_{i+1} = f(\mathbf{x}_i; \mu)$ for all $\tilde{\mu}$ in some neighbourhood of μ.

- In order to describe structural stability more generally we need to think about spaces of functions. For simplicity we consider only phase spaces $\mathcal{X}$ that are compact. We begin in one dimension.

Definition 4.3. For any $k \ge 0$, the C^k*-norm* of a C^k function $f : [a, b] \to \mathbb{R}$ is

$$\|f\|_{C^k} = \|f\|_\infty + \|f'\|_\infty + \cdots + \|f^{(k)}\|_\infty \,. \tag{4.6}$$

- The space of C^k functions $f : [a, b] \to \mathbb{R}$ together with the C^k-norm is a Banach space.

Definition 4.4. Let $f : [a, b] \to [a, b]$ be C^k. The map

$$x \mapsto f(x), \tag{4.7}$$

is said to be C^k *structurally stable* (or *structurally stable in the* C^k *topology*) if there exists $\delta > 0$ such that

$$x \mapsto f(x) + g(x), \tag{4.8}$$

is conjugate to (4.7) for every C^k g on $[a, b]$ with $\|g\|_{C^k} < \delta$.

- For example, it can be shown that (4.1) is C^1 structurally stable, but, as we found above, (4.1) is not C^0 structurally stable.

- The same ideas work in higher dimensions, but it is more complicated to write (4.6) because the $i^{\rm th}$ derivative of a function $f : \mathbb{R}^N \to \mathbb{R}^N$ is an $(i + 1)$-order tensor.

- In brief, a C^k map f on a smooth compact manifold $\mathcal{X}$ is C^k structurally stable if there exists $\delta > 0$ such that every $\tilde{f} \in B_\delta(f)$ (in the space of C^k maps on $\mathcal{X}$ equipped with the C^k topology) is conjugate to f.

- Finally we provide a technical but important result that provides an equivalence between the seeming disparate notions of structural stability and hyperbolicity. This was first proved in [6], see also [7].

Definition 4.5. A C^1 map f on a smooth manifold $\mathcal{X}$ is said to be *Axiom A* if the non-wandering set[5] of f is compact and hyperbolic[6] and the set of periodic points of f is dense in the non-wandering set.

Theorem 4.1. *A* C^1 *map* f *on a smooth compact manifold* $\mathcal{X}$ *is* C^1 *structurally stable if and only if it is Axiom A.*

5 Bifurcations

- As we saw in Part VI, bifurcations are critical parameter values at which structural stability is lost.

- Here we study codimension-one bifurcations of fixed points. These are described in [8, 9, 10] and in more detail in [11].

Definition 5.1. A fixed point $\mathbf{x}^*$ of (1.1) is said to be *hyperbolic* if no eigenvalue of $Df(\mathbf{x}^*)$ has modulus 1.

- Non-hyperbolicity occurs if $Df(\mathbf{x}^*)$ has (i) an eigenvalue 1, (ii) an eigenvalue -1, or (iii) eigenvalues $\mathrm{e}^{\pm \mathrm{i}\phi}$, where $\phi \in (0, \pi)$. These correspond to (i) saddle-node bifurcations, (ii) period-doubling (or flip) bifurcations, and (iii) Neimark-Sacker bifurcations.

[5]The *non-wandering set* is all points $\mathbf{x} \in \mathcal{X}$ with the property that for all $\varepsilon > 0$ and all $m \ge 0$ there exists $n \ge m$ such that $f^n(\mathbf{x}) \in B_\varepsilon(\mathbf{x})$. That is, the forward orbit of $\mathbf{x}$ repeatedly comes arbitrarily close to $\mathbf{x}$ (it doesn't wander away).

[6]This essentially means that the non-wandering set has no centre manifold.

- We now describe these, in order, in the lowest possible number of dimensions. In higher dimensions the bifurcations behave in the same way (on extended centre manifolds).
- First, consider the one-dimensional map

$$x \mapsto f(x;\mu), \tag{5.1}$$

where $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

Theorem 5.1. *Consider* (5.1) *where f is C^k ($k \geq 2$). Suppose*

i) $f(0;0) = 0$ *($x = 0$ is a fixed point when $\mu = 0$),*

ii) $\frac{\partial f}{\partial x}(0;0) = 1$ *(the associated stability multiplier is 1),*

iii) $\frac{\partial f}{\partial \mu}(0;0) \neq 0$ *(transversality condition),*

iv) $\frac{\partial^2 f}{\partial x^2}(0;0) \neq 0$ *(non-degeneracy condition).*

Then there exists $\delta > 0$ and a unique C^k function $\xi : [-\delta, \delta] \to \mathbb{R}$ with

$$\begin{aligned} \xi(0) &= 0, \\ \xi'(0) &= 0, \\ \xi''(0) &= -\frac{\frac{\partial^2 f}{\partial x^2}(0;0)}{\frac{\partial f}{\partial \mu}(0;0)}, \end{aligned}$$

such that $f(x, \xi(x)) = x$ for all $x \in [-\delta, \delta]$.

- If (5.1) satisfies the conditions of Theorem 5.1, we say that (5.1) has a *saddle-node bifurcation* at $\mu = 0$. Here two fixed points (one stable, one unstable) collide and annihilate.

Exercise 5.1. Consider the one-dimensional map

$$f(x;\gamma) = \frac{1}{x} + \gamma x^2. \tag{5.2}$$

Find $\gamma > 0$ at which (5.2) has a saddle-node bifurcation. Hint: Either (i) solve $f(x;\gamma) = x$ and $\frac{\partial f}{\partial x}(x;\gamma) = 1$ simultaneously, or (ii) rearrange $f(x;\gamma) = x$ as $\gamma = h(x)$ and solve $h'(x) = 0$.

Exercise 5.2. Show that the two-dimensional map

$$f(x, y; \alpha, \beta) = \begin{bmatrix} \alpha - \beta y - x^2 \\ x \end{bmatrix}, \tag{5.3}$$

has saddle-node bifurcations along $\alpha = -\frac{(\beta+1)^2}{4}$.

Theorem 5.2. *Consider* (5.1) *where f is C^k ($k \geq 3$). Suppose*

i) $f(0;0) = 0$ *($x = 0$ is a fixed point when $\mu = 0$),*

ii) $\frac{\partial f}{\partial x}(0;0) = -1$ *(the associated stability multiplier is -1),*

iii) $\alpha = \left(\frac{\partial^2 f}{\partial \mu \partial x} + \frac{1}{2} \frac{\partial f}{\partial \mu} \frac{\partial^2 f}{\partial x^2} \right) \Big|_{(x;\mu)=(0,0)} \neq 0$ *(transversality condition),*

iv) $\beta = \left(\frac{1}{2} \left(\frac{\partial^2 f}{\partial x^2} \right)^2 + \frac{1}{3} \frac{\partial f^3}{\partial x^3} \right) \Big|_{(x;\mu)=(0,0)} \neq 0$ *(non-degeneracy condition).*

Then there exists $\delta > 0$ and a unique C^{k-1} function $\xi : [-\delta, \delta] \to \mathbb{R}$ with

$$\begin{aligned} \xi(0) &= 0, \\ \xi'(0) &= 0, \\ \xi''(0) &= -\frac{\beta}{\alpha}, \end{aligned}$$

such that $f^2(x, \xi(x)) = x$ for all $x \in [-\delta, \delta]$.

- Note that in the last line of Theorem 5.2, f^2 refers to the second iterate of f (not the square of f).
- If (5.1) satisfies the conditions of Theorem 5.2, we say that (5.1) has a *period-doubling bifurcation* at $\mu = 0$. Here a fixed point changes stability and a period-2 solution is created.

Exercise 5.3. Consider

$$f(x;\eta) = x^2 + \eta. \tag{5.4}$$

i) Compute fixed points and period-2 solutions of (5.4). Hint: These can both be expressed as the roots of a quadratic equation.

ii) Show that (5.4) has a period-doubling bifurcation at $\eta = \frac{3}{4}$.

- Lastly, consider the two-dimensional map

$$\mathbf{x} \mapsto f(\mathbf{x};\mu), \tag{5.5}$$

where $f : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$.

Theorem 5.3. *Consider* (5.5) *where f is C^k ($k \geq 4$). Suppose* *i)* $f(\mathbf{0};\mu) = \mathbf{0}$ *for all* μ *in a neighbourhood of* 0 *(*$\mathbf{x} = \mathbf{0}$ *is a fixed point for small* μ*),*

ii) $Df(\mathbf{0};\mu)$ *has eigenvalues* $r(\mu)\mathrm{e}^{\pm \mathrm{i}\phi(\mu)}$ *with* $r(0) = 1$ *and* $\mathrm{e}^{in\phi(0)} \neq 1$ *for* $n = 1,2,3,4$ *(at* $\mu = 0$ *the associated stability multipliers have modulus* 1 *and are not* strongly resonant*),*

iii) $r'(0) \neq 0$ *(transversality condition),*

iv) $\alpha \neq 0$ *where* α *is the* first Lyapunov coefficient[7] *(non-degeneracy condition).*

Then (5.5) *has an invariant topological circle, of size asymptotically proportional to* $\sqrt{|\mu|}$*, emanating from* $\mathbf{x} = \mathbf{0}$ *for either* $\mu < 0$ *or* $\mu > 0$*.*

- If (5.5) satisfies the conditions of Theorem 5.3, we say that (5.5) has a *Neimark-Sacker bifurcation* at $\mu = 0$. Here a fixed point changes stability and an invariant circle is created on which the dynamics may be quasiperiodic or weakly resonant.

6 A first look at the logistic family

- The one-parameter family of maps

$$f_a(x) = ax(1-x), \tag{6.1}$$

where $a \in [0,4]$, is known as the *logistic family.* For all $a \in [0,4]$, f_a is an S-unimodal map on $[0,1]$ and so has a unique attractor.

- Fig. 2 shows a bifurcation diagram of (6.1) illustrating this attractor.
- A transcritical bifurcation occurs at $a = 1$. Period-doubling bifurcations occur at $a_1, a_2, \ldots$ where the first few values are given by:

k	a_k	period created	F_k
1	3	2	
2	$3.4494\ldots$	4	$4.7514\ldots$
3	$3.5440\ldots$	8	$4.6562\ldots$
4	$3.5644\ldots$	16	$4.6683\ldots$
5	$3.5687\ldots$	32	$4.6686\ldots$

- This is known as a *period-doubling cascade.* Such cascades are observed in many dynamical systems and provide a simple 'route to chaos'.

- In the above table

$$F_k = \frac{a_k - a_{k-1}}{a_{k+1} - a_k}. \tag{6.2}$$

The limit

$$\lim_{k\to\infty} F_k = 4.6692\ldots \tag{6.3}$$

known as Feigenbaum's constant, describes the asymptotic rate at which the period-doubling bifurcations occur. Importantly, this constant is universal in the sense that all S-unimodal maps exhibit period-doubling cascades at this rate.

Exercise 6.1. Use MATLAB to show that f_4 appears to exhibit sensitive dependence on initial conditions.

Exercise 6.2. Consider the alternate quadratic family

$$g_b(y) = y^2 - b. \tag{6.4}$$

where $b \in \left[-\frac{1}{4}, 2\right]$. Here we show that (6.4) is equivalent to (6.1) for $a \in [1,4]$.

i) Show that $y^* = \frac{1}{2} + \sqrt{\frac{1}{4} + b}$ is a fixed point of (6.4).

ii) Show that the affine coordinate change $x = \frac{y^* - y}{2y^*}$ converts (6.4) into (6.1) with $a = 2y^* \in [1,4]$.

7 Lyapunov exponents

- Lyapunov exponents measure the asymptotic rate at which nearby orbits converge or diverge. They can be defined for both ODEs and maps. Here we consider a smooth map $f : \mathbb{R}^N \to \mathbb{R}^N$.
- Let $\mathbf{x} \in \mathbb{R}^N$ be a point, let $\mathbf{n} \in \mathbb{R}^N$ with $\|\mathbf{n}\| = 1$ be a unit vector, and let $\delta > 0$ be small. Then $\mathbf{x} + \delta\mathbf{n}$ represents the perturbation from $\mathbf{x}$ in direction $\mathbf{n}$ of magnitude δ. We are interested in the difference between $f^n(\mathbf{x} + \delta\mathbf{n})$ and $f^n(\mathbf{x})$ for large values of n.
- It is reasonable to assume that

$$\|f^n(\mathbf{x} + \delta\mathbf{n}) - f^n(\mathbf{x})\| \sim \delta \mathrm{e}^{\chi n},$$

for some $\chi \in \mathbb{R}$. This can be rearranged to produce

$$\chi \sim \frac{1}{n} \ln\left(\frac{1}{\delta}\|f^n(\mathbf{x} + \delta\mathbf{n}) - f^n(\mathbf{x})\|\right). \tag{7.1}$$

[7] See for instance [11] or *http://www.scholarpedia.org/article/Neimark-Sacker_bifurcation.*

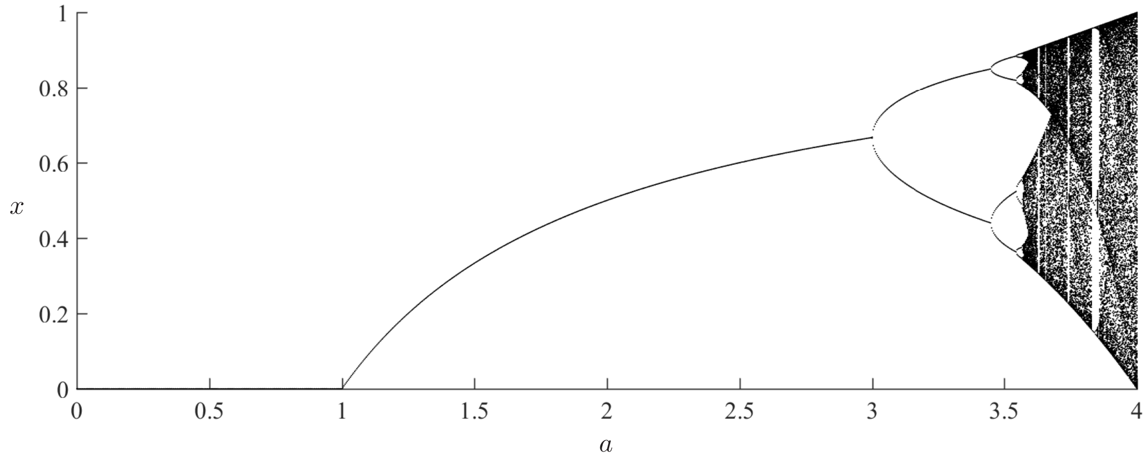

Figure 2: A bifurcation diagram of the logistic family (6.1).

This motivates defining χ by taking the limits $\delta \to 0$ and $n \to \infty$ in (7.1) giving

$$\chi = \lim_{n\to\infty} \frac{1}{n} \ln(\|(Df^n)(\mathbf{x})\mathbf{n}\|), \qquad (7.2)$$

assuming the limit in (7.2) exists.

- A quite technical theorem of Oseledet[8] tells us that, under certain conditions that in practice usually hold, the limit in (7.2) does exist and for fixed $\mathbf{x}$ and all $\mathbf{n}$ can at take most N values (these are the *Lyapunov exponents* of f at $\mathbf{x}$).
- The *maximal Lyapunov exponent*, call it $\chi_{\max}$, is most important because (7.2) takes this value for almost all $\mathbf{n}$.
- If $f^n(\mathbf{x})$ converges to a hyperbolic, asymptotically stable fixed point $\mathbf{x}^*$ of f, typically it will do so asymptotic to the slowest eigendirection. In this case $\chi_{\max} < 0$ and is equal to the natural log of the modulus of the eigenvalue of $(Df)(\mathbf{x}^*)$ with largest modulus.
- If f exhibits sensitive dependence on initial conditions, then we usually have $\chi_{\max} > 0$. Compared with other aspects of chaos, it is relatively straight-forward to compute maximal Lyapunov exponents numerically (although some diligence is often required to calculate them to more than two significant figures). For this reason, to decide whether or not a given dynamical system is chaotic, it is common to simply look at the sign of a numerically computed maximal Lyapunov exponent.

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