Piecewise-linear maps: where do they come from and what do they do?

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  - piecewise-linear
  - with two pieces
  - continuous

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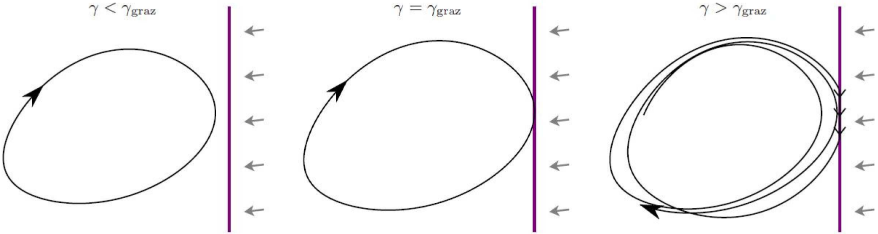
$$= \begin{cases} \begin{bmatrix} a & 1 \\ b & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & x_1 \leq 0 \\ \begin{bmatrix} -a & 1 \\ b & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & x_1 \geq 0 \end{cases}$$

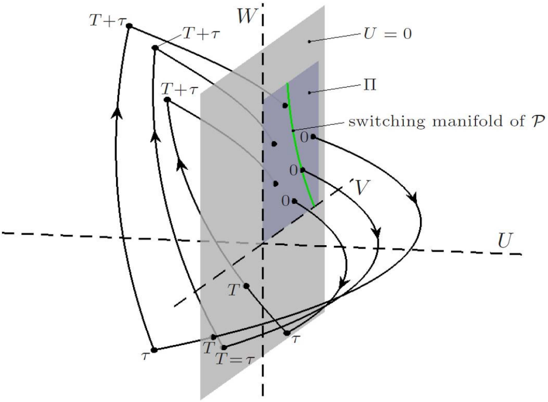
Piecewise-smooth maps as Poincaré maps

1) Grazing-sliding bifurcations



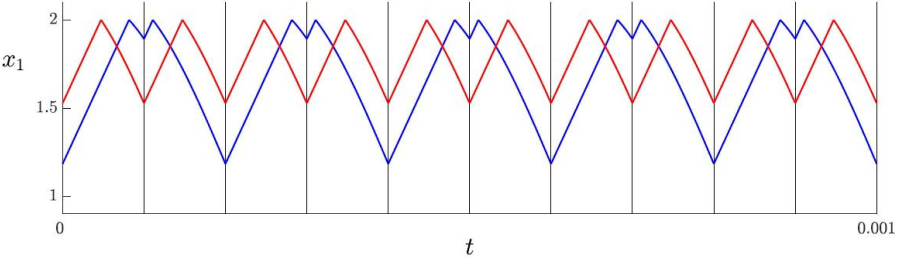
- Get a zero eigenvalue in one piece of the map due to the sliding motion
  - di Bernardo, Kowalczyk & Nordmark, 2002

2) Event collisions for ODEs with time-delayed switching



- Seiber, 2006

3) Corner collisions



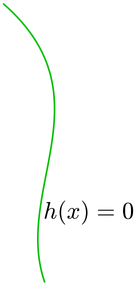
- Often get a non-invertible map due to the switching
  - di Bernardo, Budd & Champneys, 2001

Border-collision bifurcations

- Consider a continuous piecewise-smooth map

$$x \mapsto \begin{cases} f_L(x;\mu), & h(x) \le 0, \\ f_R(x;\mu), & h(x) \ge 0, \end{cases}$$

where $\mu \in \mathbb{R}$ is a parameter.



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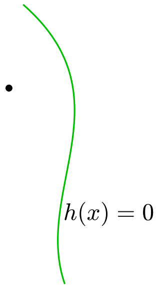

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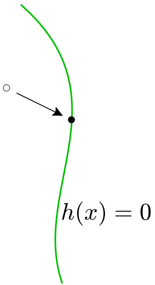

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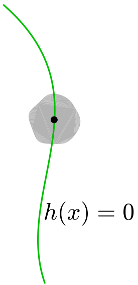
- Locally can approximate as
$$x \mapsto \begin{cases} A_L x + b\mu, & x_1 \le 0, \\ A_R x + b\mu, & x_1 \ge 0, \end{cases}$$
where A_L and A_R differ only in their first columns.



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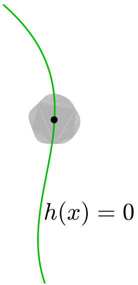

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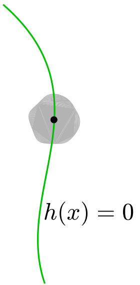
- Often the truncation to piecewise-linear is justified.



Many different dynamical transitions

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North-Holland

PHYSICA D

Border-collision bifurcations including "period two to period three" for piecewise smooth systems*

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[c]*Department of Mathematics, University of Maryland, College Park, MD 20742, USA*

The border-collision normal form

- If (A_L, e_1) is observable, equivalently if A_L has no eigenvector v with $v_1 = 0$, then

$$x \mapsto \begin{cases} A_L x + b\mu, & x_1 \le 0, \\ A_R x + b\mu, & x_1 \ge 0, \end{cases}$$

can be transformed to

$$x \mapsto \begin{cases} C_L x + e_1\mu, & x_1 \le 0, \\ C_R x + e_1\mu, & x_1 \ge 0, \end{cases}$$

with companion matrices:

$$C_L = \begin{bmatrix} p_1 & 1 & & \\ p_2 & & \ddots & \\ \vdots & & & 1 \\ p_n & & & \end{bmatrix}, \qquad C_R = \begin{bmatrix} q_1 & 1 & & \\ q_2 & & \ddots & \\ \vdots & & & 1 \\ q_n & & & \end{bmatrix}, \qquad e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

The border-collision normal form

- In two dimensions

$$x \mapsto \begin{cases} \begin{bmatrix} \tau_L & 1 \\ -\delta_L & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mu, & x_1 \le 0, \\ \begin{bmatrix} \tau_R & 1 \\ -\delta_R & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mu, & x_1 \ge 0. \end{cases}$$

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- If $\tau_L = -\tau_R = a$, $\delta_L = \delta_R = -b$, and $\mu = 1$, get the Lozi family

$$x \mapsto \begin{cases} \begin{bmatrix} a & 1 \\ b & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & x_1 \le 0, \\ \begin{bmatrix} -a & 1 \\ b & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & x_1 \ge 0. \end{cases}$$

Outline

1) A sausage-string structure for periodicity regions.
2) Homoclinic corners and infinitely many attractors.
3) Stability of boundary fixed points.

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Periodicity regions

$$x \mapsto \begin{cases} A_L x + b\mu, & x_1 \leq 0, \\ A_R x + b\mu, & x_1 \geq 0. \end{cases}$$

$$A_L = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0.15 & 0 & 0 \end{bmatrix}, \quad A_R = \begin{bmatrix} -1.5 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

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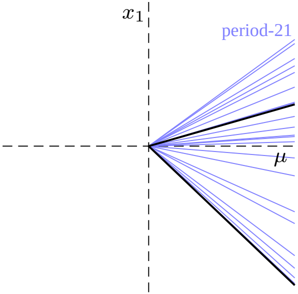

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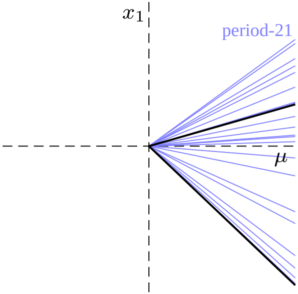

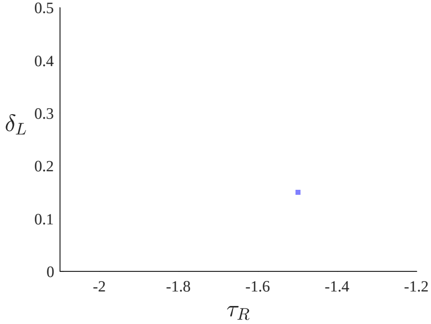

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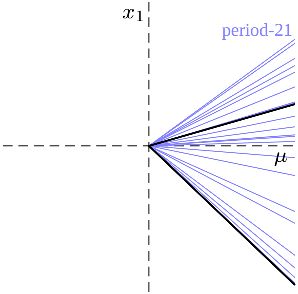

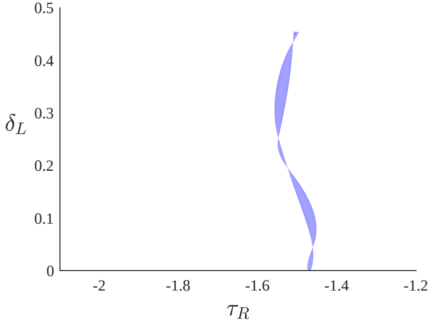

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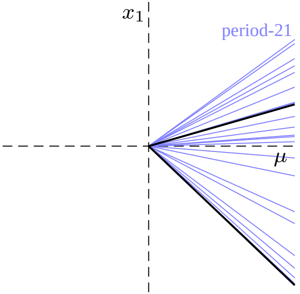

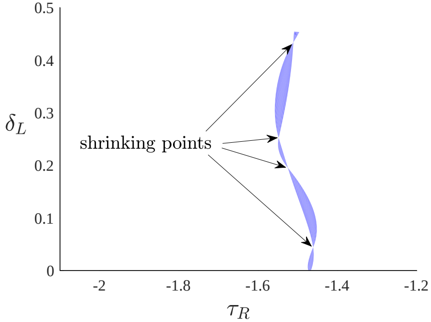

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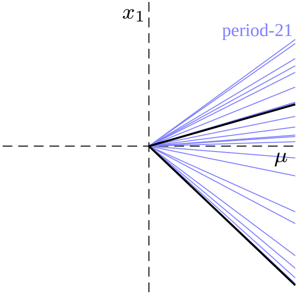
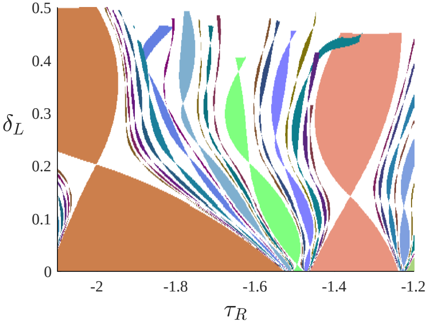


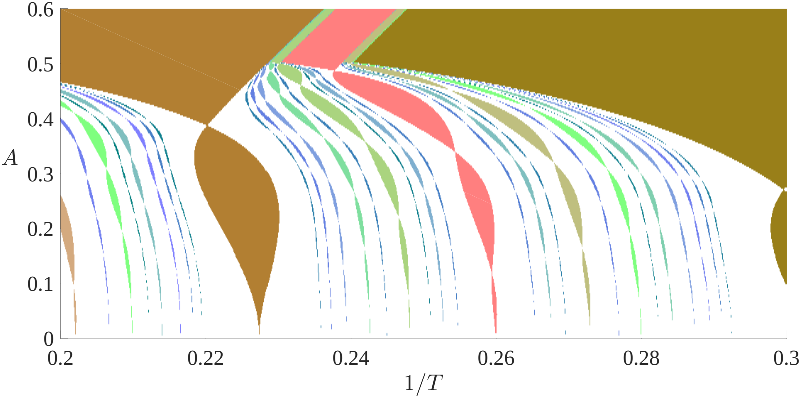


Integrate-and-fire neurons

$$\text{potential:} \qquad \dot{V} = -V + I - F(t) \qquad (I = 1.5)$$

$$\text{reset law:} \qquad V(t) = 1 \;\to\; V(t) = 0$$

$$\text{square-wave forcing:} \qquad F(t) = A\,\mathrm{sgn}\Big[\sin\Big(\frac{2\pi t}{T}\Big)\Big]$$

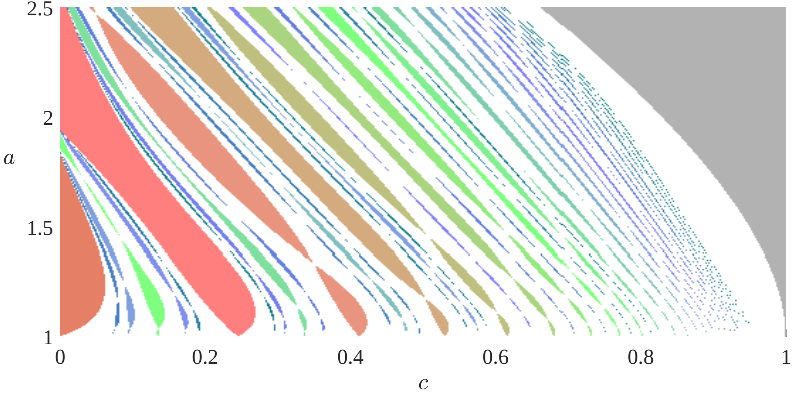


- Tiesinga, 2002

Business cycle dynamics

$$\text{income:} \qquad Y_t = cY_{t-1} + I_t$$

$$\text{net investment:} \qquad I_t = \max[a(Y_{t-1} - Y_{t-2}), -arY_{t-2}]$$



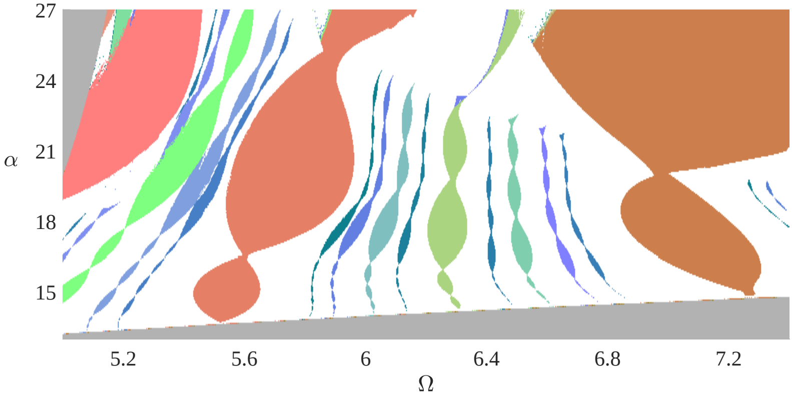
- Sushko, Gardini & Puu, 2004

DC/DC power converters

$$\dot{y}_k = \lambda_k(y_k - K_F(t)), \quad k = 1,2$$

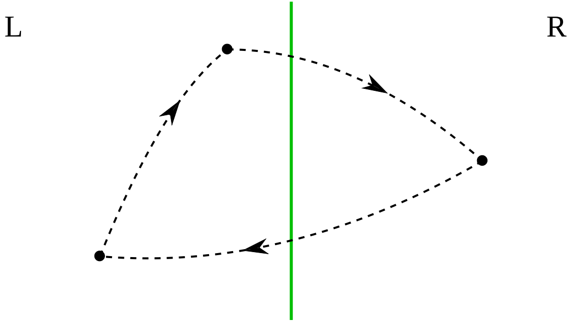
$$K_F(t) = \mathbf{1}_{\xi > \frac{q}{\alpha\Omega}(t - \lfloor t \rfloor)}$$

$$\xi = y_1(\lfloor t \rfloor) - \theta y_2(\lfloor t \rfloor) + \frac{q}{2\Omega}$$

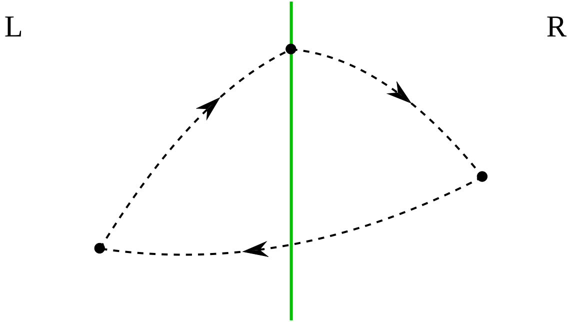


- Zhusubaliyev & Mosekilde, 2008

- An LLR-cycle:



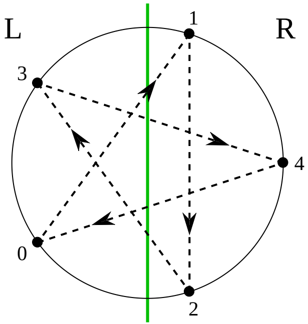
- The left and right boundaries of the periodicity regions are border-collision bifurcations where one point of an $\mathcal{S}$-cycle lies on the switching manifold.

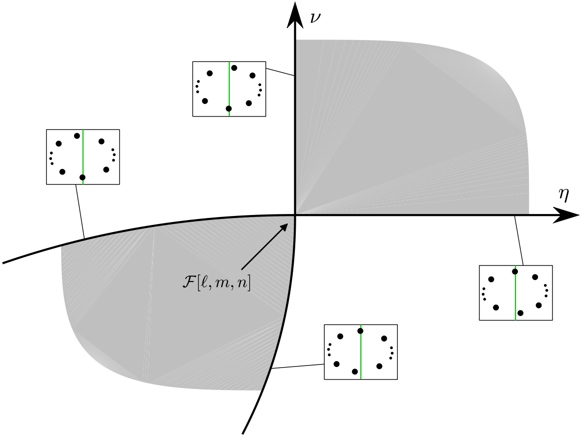


- Shrinking points are where an $\mathcal{S}$-cycle has two points on the switching manifold, where $\mathcal{S} = \mathcal{F}[\ell, m, n]$ corresponds to rigid rotation on a circle:

$$n - \text{period}$$
$$\frac{m}{n} - \text{rotation number}$$
$$\ell - \text{number of L's}$$

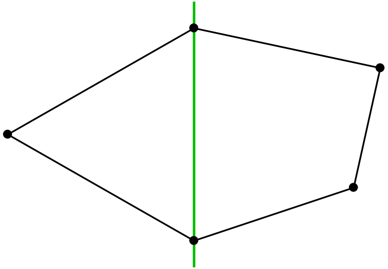
- For example, $\mathcal{F}[2,2,5] = LRRLR \quad \rightarrow$





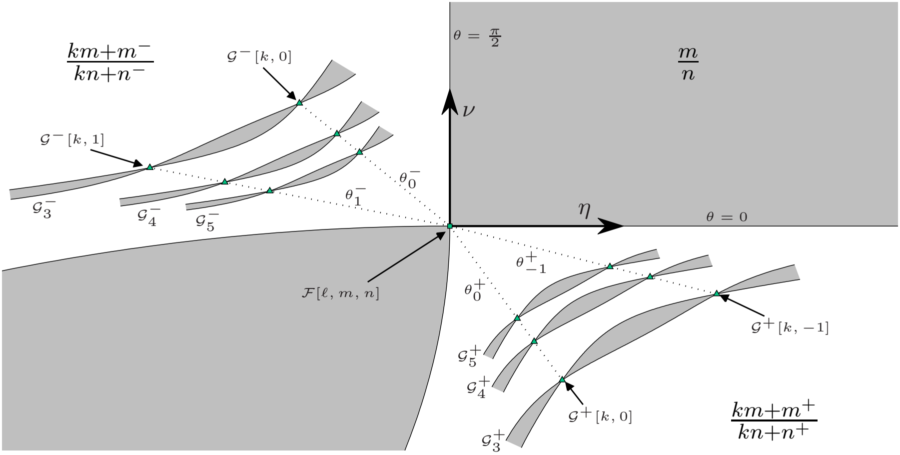
- Four distinct curves of border-collision bifurcations.
- These admit a nice combinatorical characterisation in terms of ℓ, m, and n.
  - S & Meiss, *Nonlinearity*, 2009 & 2010

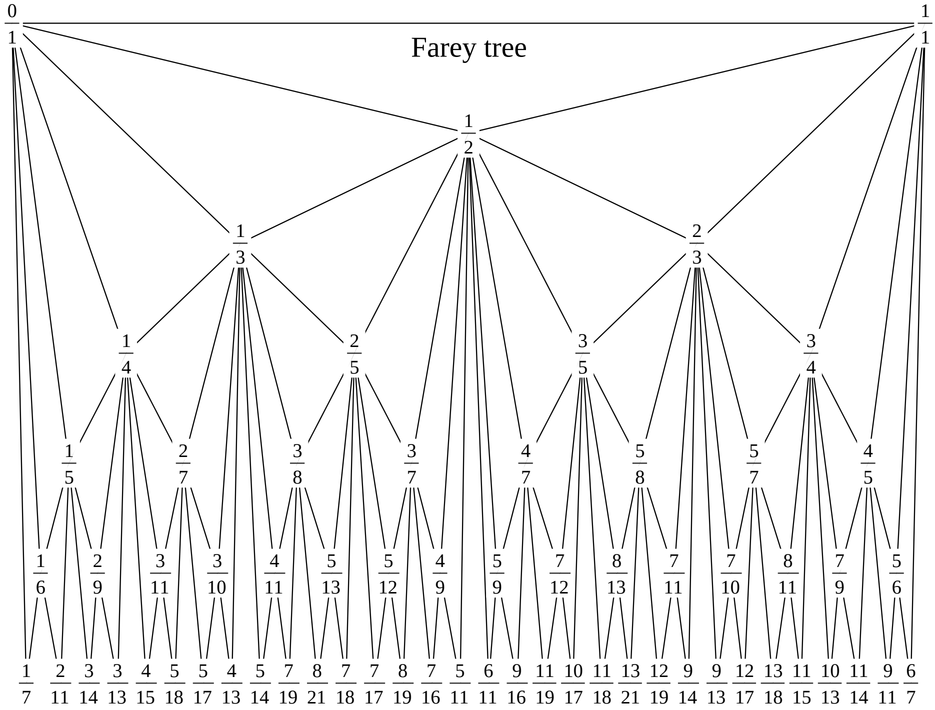
- At a shrinking point there exists an invariant polygon.



- Near the shrinking point this persists as a one-dimensional slow manifold.

- Near a shrinking point there are two primary sequences of periodicity regions.




Farey tree
0/1
1/1
1/2
1/3
2/3
1/4
2/5
3/5
3/4
1/5
2/7
3/8
3/7
4/7
5/8
5/7
4/5
1/6
2/9
3/11
3/10
4/11
5/13
5/12
4/9
5/9
7/12
8/13
7/11
7/10
8/11
7/9
5/6
1/7
2/11
3/14
3/13
4/15
5/18
5/17
4/13
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7/19
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6/11
9/16
11/19
10/17
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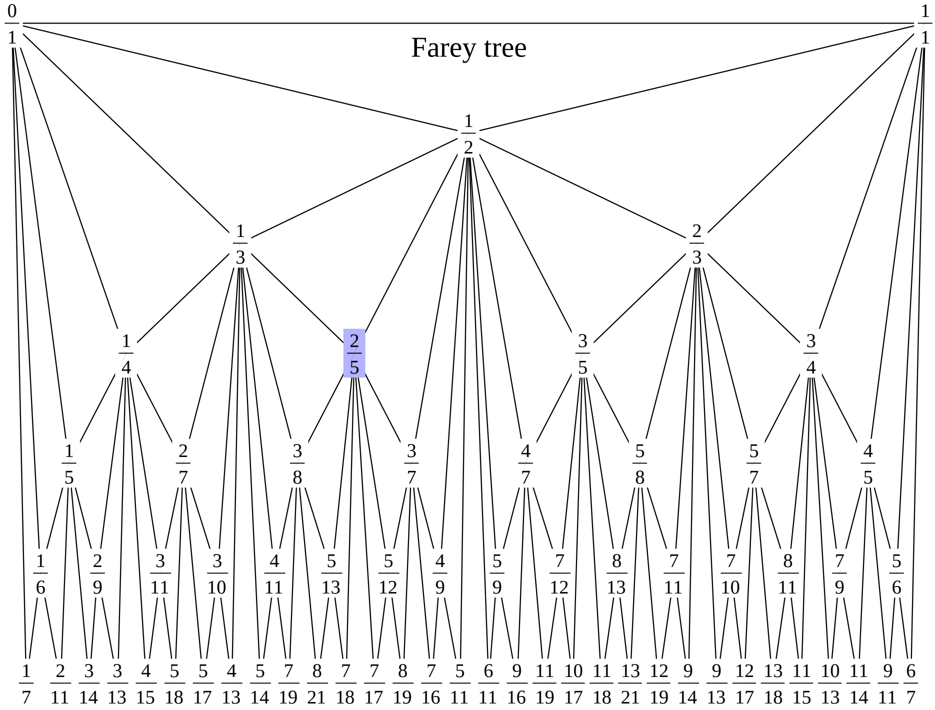

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0/1
1/1
1/2
1/3
2/3
1/4
2/5
3/5
3/4
1/5
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4/7
5/8
5/7
4/5
1/6
2/9
3/11
3/10
4/11
5/13
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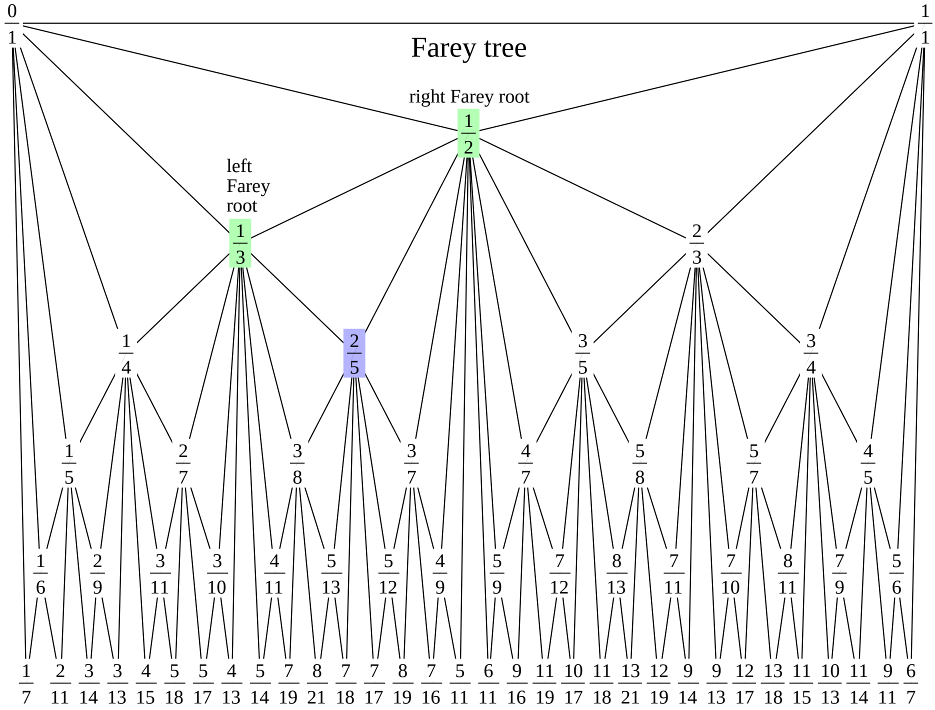

Farey tree
right Farey root
left Farey root
0/1
1/1
1/2
1/3
2/3
1/4
2/5
3/5
3/4
1/5
2/7
3/8
3/7
4/7
5/8
5/7
4/5
1/6
2/9
3/11
3/10
4/11
5/13
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4/15
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10/17
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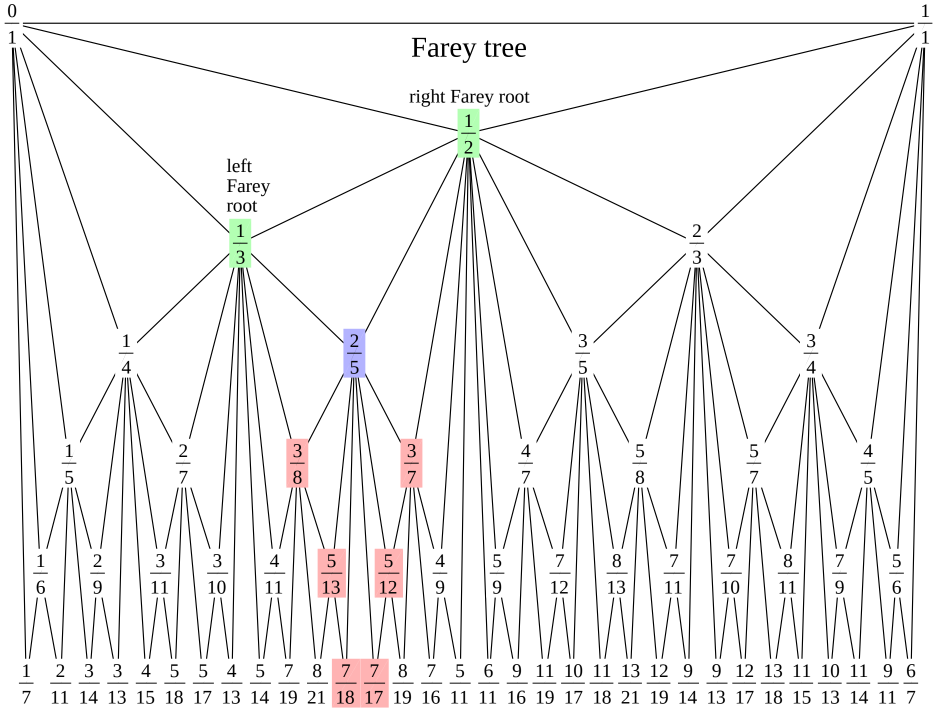
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1/4
2/5
3/5
3/4
1/5
2/7
3/8
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4/7
5/8
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1/6
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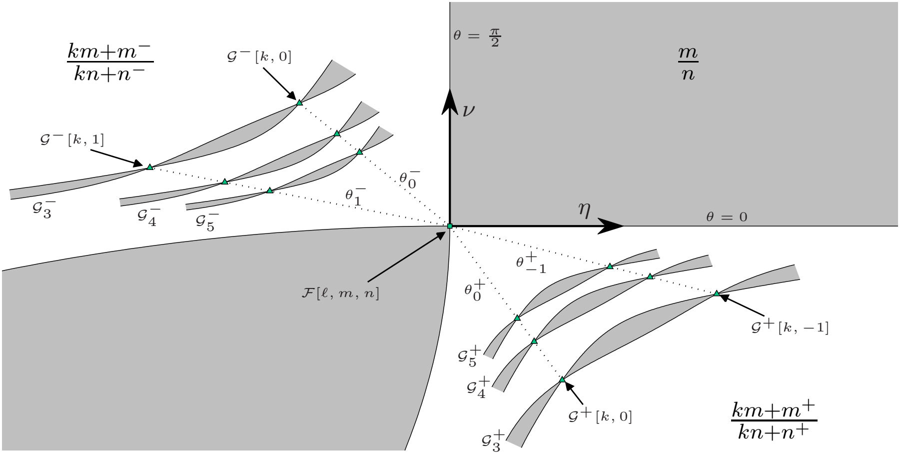
8/11
7/9
5/6
1/7
2/11
3/14
3/13
4/15
5/18
5/17
4/13
5/14
7/19
8/21
7/18
7/17
8/19
7/16
5/11
6/11
9/16
11/19
10/17
11/18
13/21
12/19
9/14
9/13
12/17
13/18
11/15
10/13
11/14
9/11
6/7

- Near a shrinking point there are two primary sequences of periodicity regions.

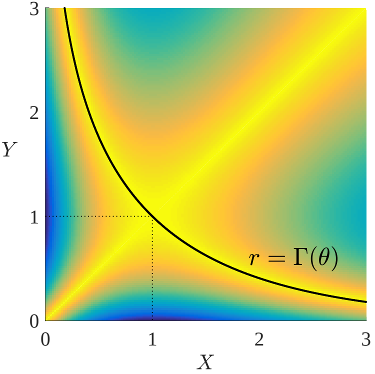


- Apply a certain linear coordinate scaling: $\eta \to X, \nu \to Y$
- Introduce polar coordinates: $X = r\cos(\theta), Y = r\sin(\theta)$

THEOREM (S, *Nonlinearity*, 2017)

Primary periodicity regions are within $\mathcal{O}\left(\frac{1}{k^2}\right)$ *of* $r = \frac{1}{k}\Gamma(\theta)$*, where* $r = \Gamma(\theta)$ *is the non-trivial solution to*

$$X\mathrm{e}^{-X} = Y\mathrm{e}^{-Y}$$



Theorem (S, *Nonlinearity*, 2017)

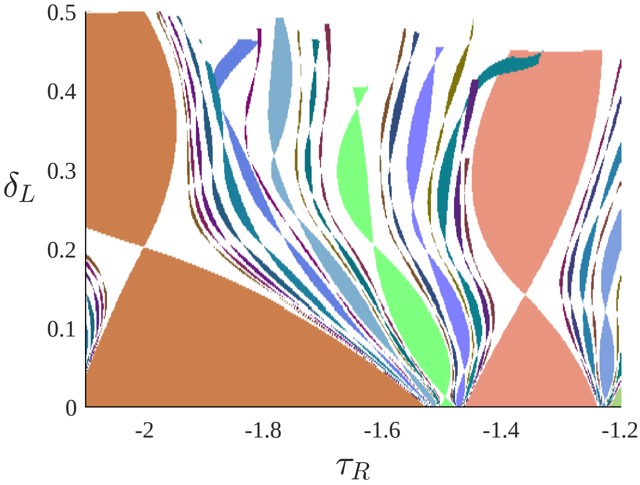
On primary periodicity regions, $\mathcal{G}^{\pm}[k, \Delta\ell]$-shrinking points exist for large k if and only if $\kappa^{\pm}_{\Delta\ell} > 0$ and are located at $\theta = \theta^{\pm}_{\Delta\ell} + \mathcal{O}\left(\frac{1}{k}\right)$ where:

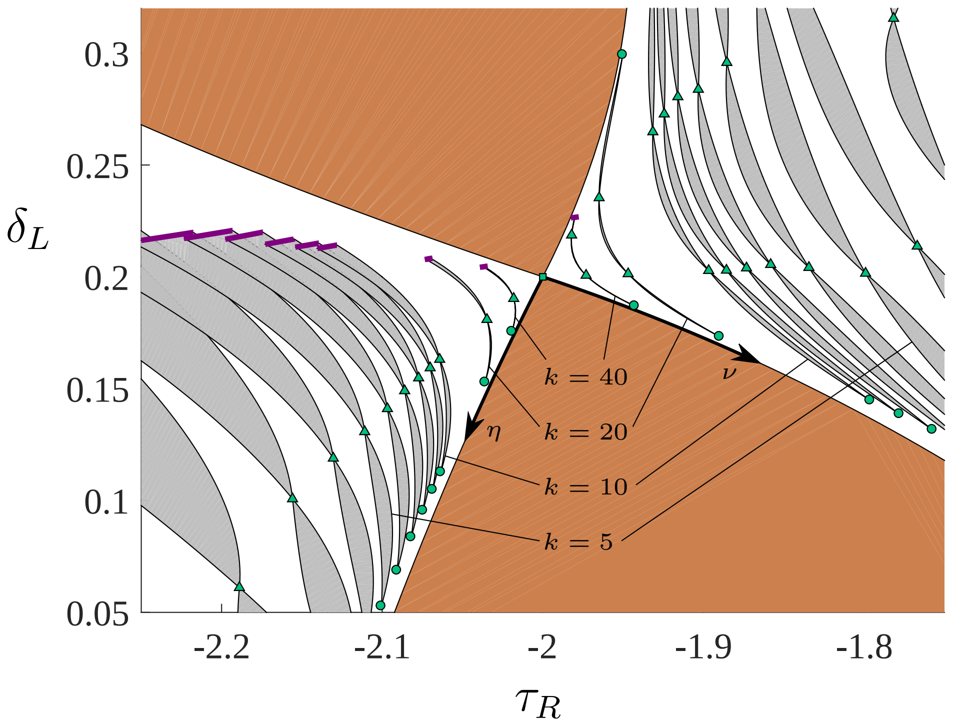
For $\Delta\ell \geq 0$,

$$\kappa^{+}_{\Delta\ell} = u_0^{\mathsf{T}} M^{\Delta\ell}_{\mathcal{S}^{\overline{\ell d}}} v_{-d}$$

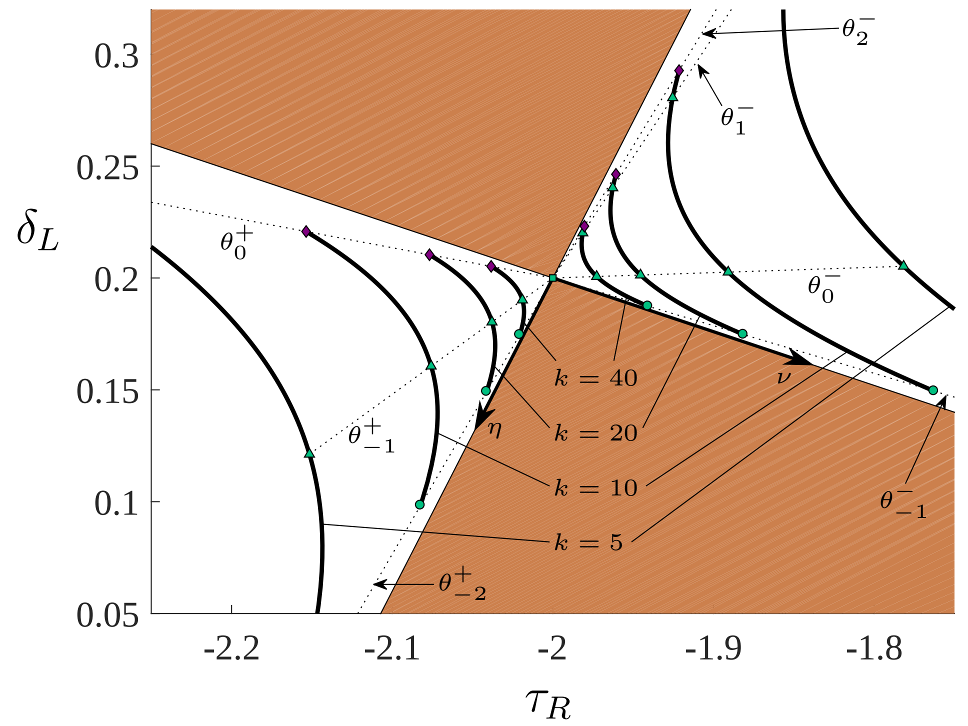
$$\theta^{+}_{\Delta\ell} = \tan^{-1}\left(\frac{t_d}{t_{-d}\left|\kappa^{+}_{\Delta\ell}\right|}\right)$$

etc.

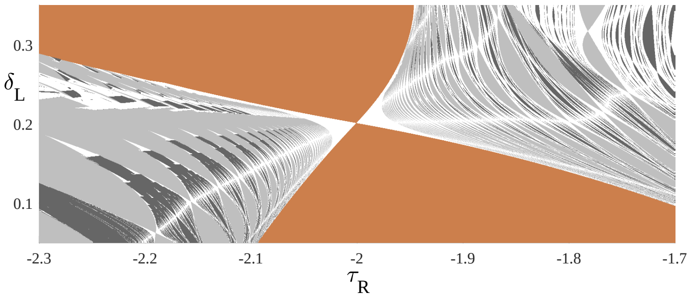

0.5
0.4
0.3
0.2
0.1
0
δ_L
-2
-1.8
-1.6
-1.4
-1.2
τ_R



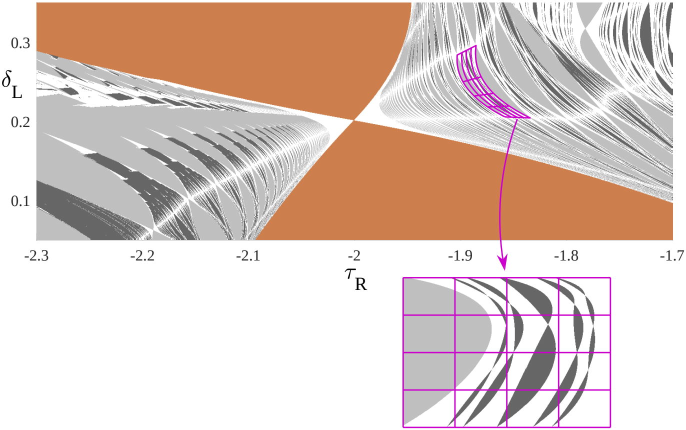
0.3
0.25
0.2
0.15
0.1
0.05
δ_L
-2.2
-2.1
-2
-1.9
-1.8
τ_R
ν
η
$k = 40$
$k = 20$
$k = 10$
$k = 5$



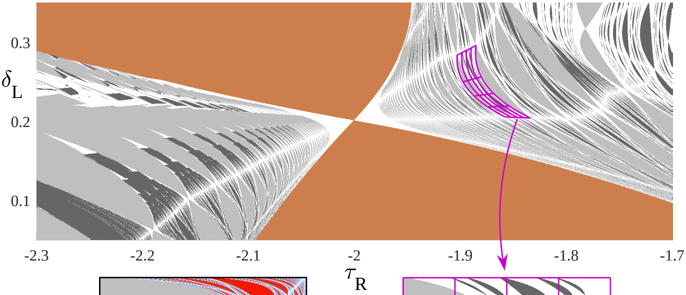
δ_L
τ_R
0.3
0.25
0.2
0.15
0.1
0.05
-2.2
-2.1
-2
-1.9
-1.8
θ_2^-
θ_1^-
θ_0^+
θ_0^-
θ_{-1}^+
θ_{-1}^-
θ_{-2}^+
ν
η
$k = 40$
$k = 20$
$k = 10$
$k = 5$

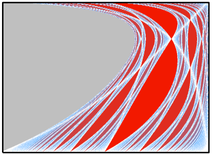


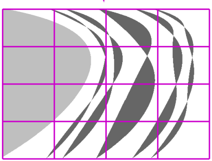
0.3
δ_L
0.2
0.1
-2.3
-2.2
-2.1
-2
-1.9
-1.8
-1.7
τ_R



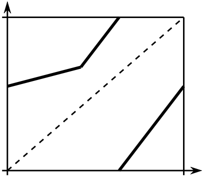
0.3
δ_L
0.2
0.1
-2.3
-2.2
-2.1
-2
-1.9
-1.8
-1.7
τ_R



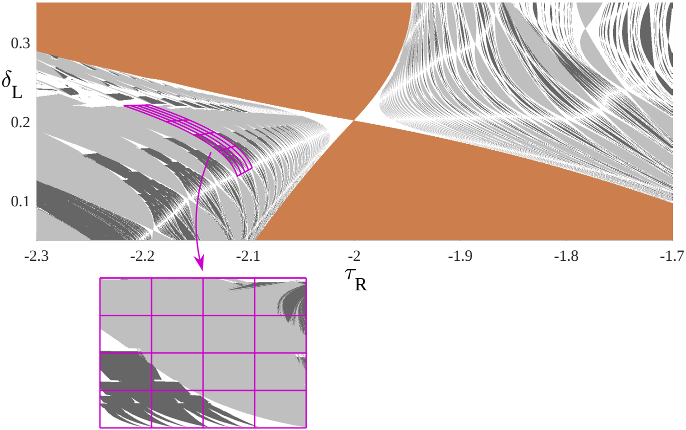


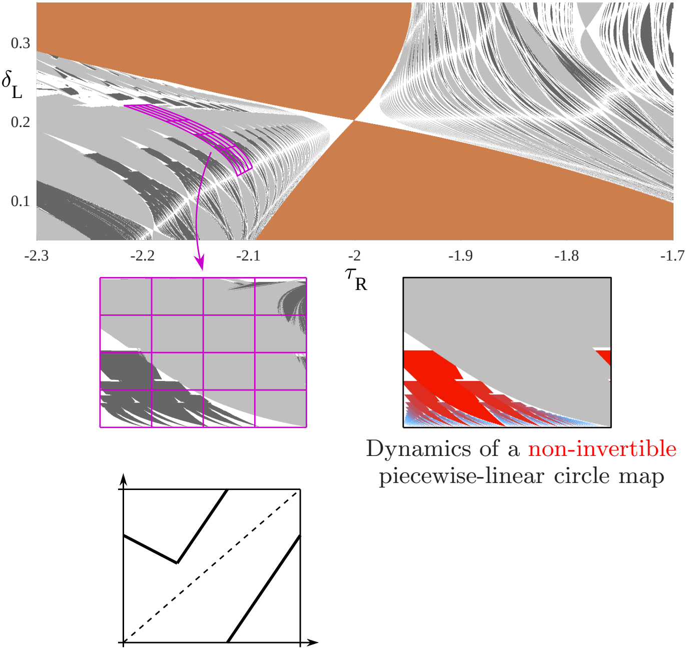


Dynamics of an invertible
piecewise-linear circle map



- S, *Nonlinearity*, 2018


0.3
δ_L
0.2
0.1
-2.3
-2.2
-2.1
-2
-1.9
-1.8
-1.7
τ_R



0.3
δ_L
0.2
0.1
-2.3
-2.2
-2.1
-2
-1.9
-1.8
-1.7
τ_R
Dynamics of a non-invertible
piecewise-linear circle map

Non-rotational periodic solutions

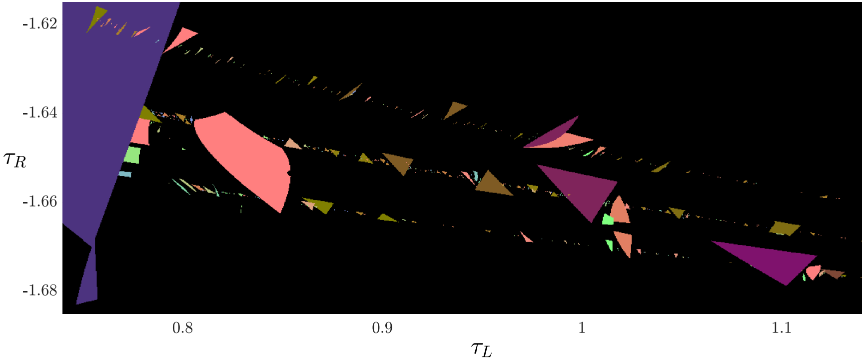
- E.g. with $\delta_L = 0.1$, $\delta_R = 1.2$,

$$x \mapsto \begin{cases} \begin{bmatrix} \tau_L & 1 \\ -\delta_L & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mu, & x_1 \leq 0, \\ \begin{bmatrix} \tau_R & 1 \\ -\delta_R & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mu, & x_1 \geq 0. \end{cases}$$

Non-rotational periodic solutions

- E.g. with $\delta_L = 0.1$, $\delta_R = 1.2$,

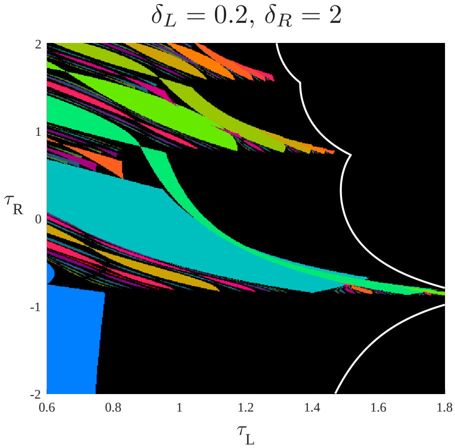
$$x \mapsto \begin{cases} \begin{bmatrix} \tau_L & 1 \\ -\delta_L & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mu, & x_1 \leq 0, \\ \begin{bmatrix} \tau_R & 1 \\ -\delta_R & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mu, & x_1 \geq 0. \end{cases}$$



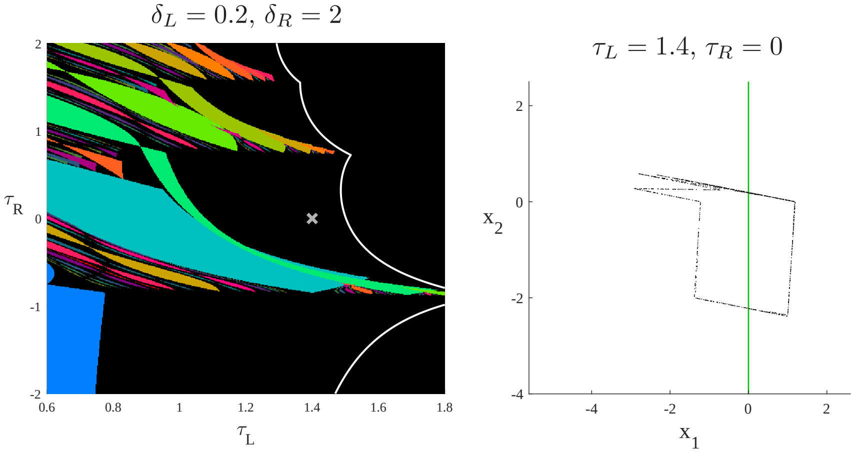
Outline

1) A sausage-string structure for periodicity regions.
2) Homoclinic corners and infinitely many attractors.
3) Stability of boundary fixed points.

Homoclinic corners

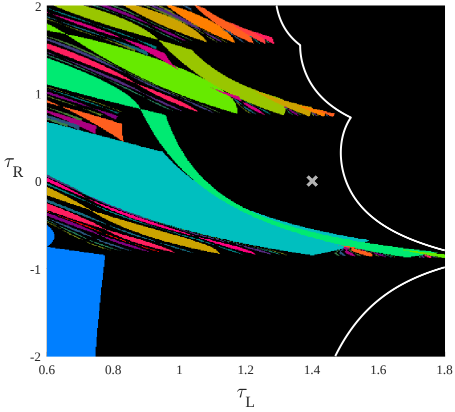


Homoclinic corners

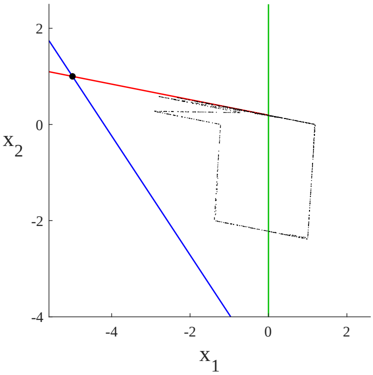


Homoclinic corners

$\delta_L = 0.2,\ \delta_R = 2$

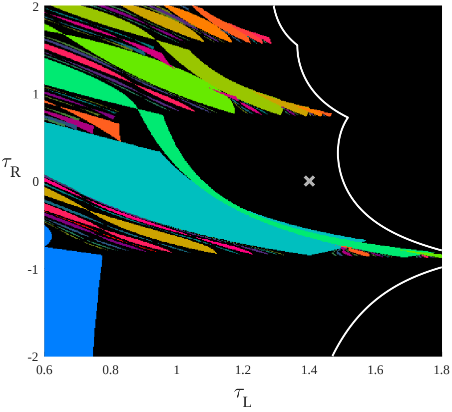


$\tau_L = 1.4,\ \tau_R = 0$

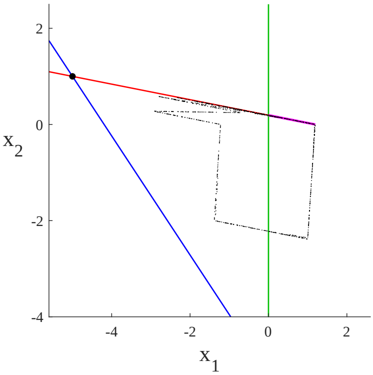


Homoclinic corners

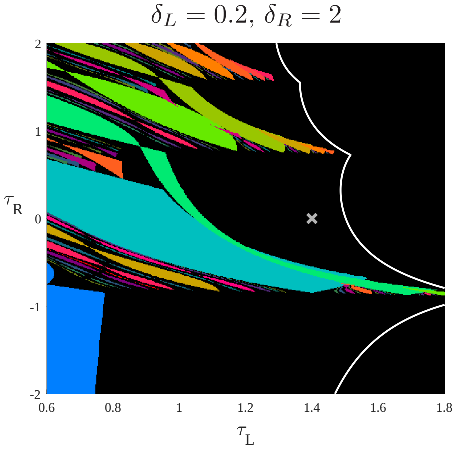
$\delta_L = 0.2,\ \delta_R = 2$

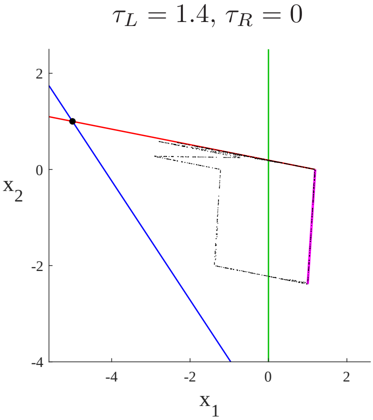


$\tau_L = 1.4,\ \tau_R = 0$



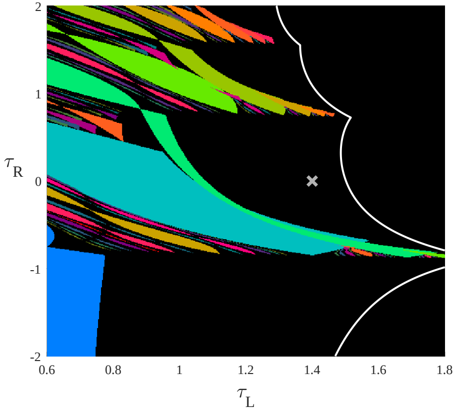
Homoclinic corners



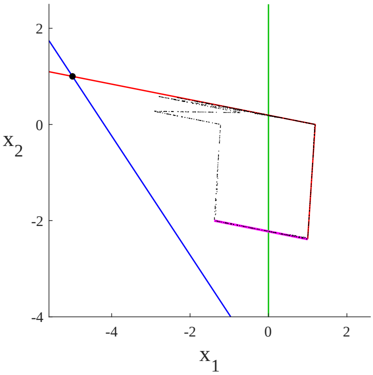


Homoclinic corners

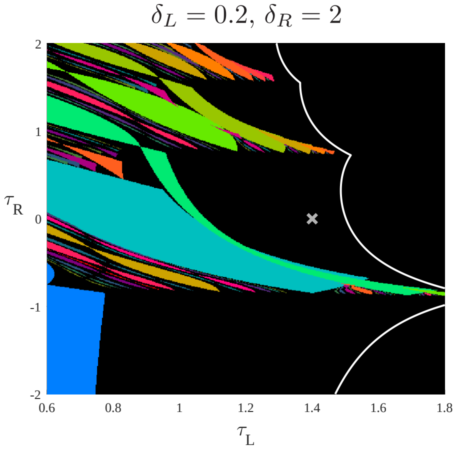
$\delta_L = 0.2,\ \delta_R = 2$

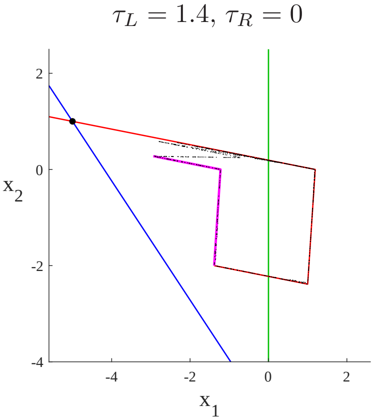


$\tau_L = 1.4,\ \tau_R = 0$

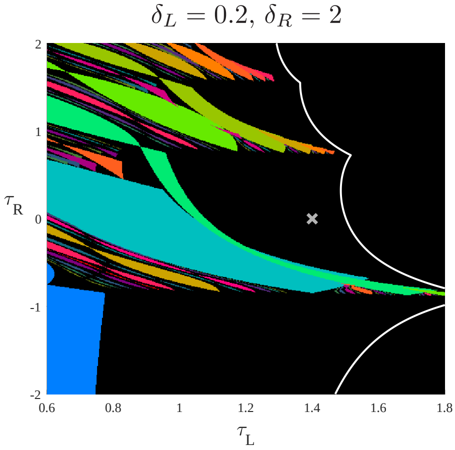


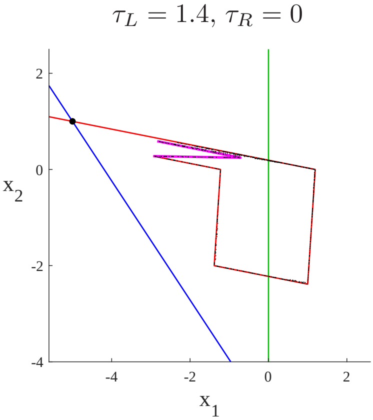
Homoclinic corners



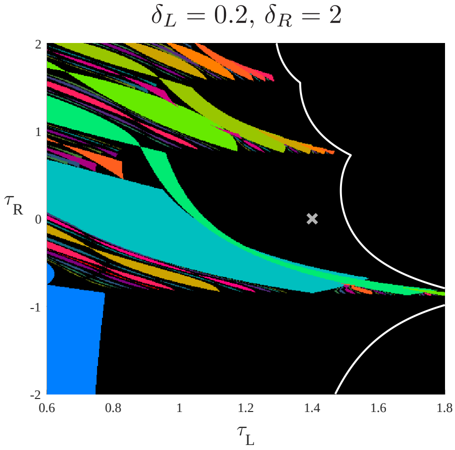


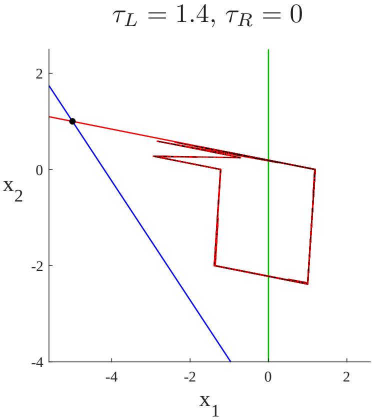
Homoclinic corners



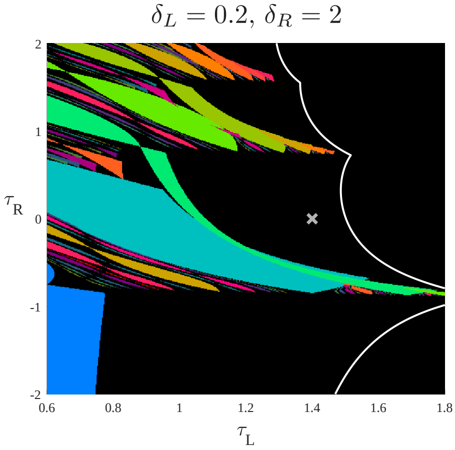


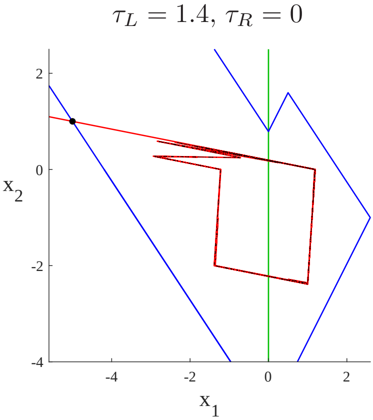
Homoclinic corners



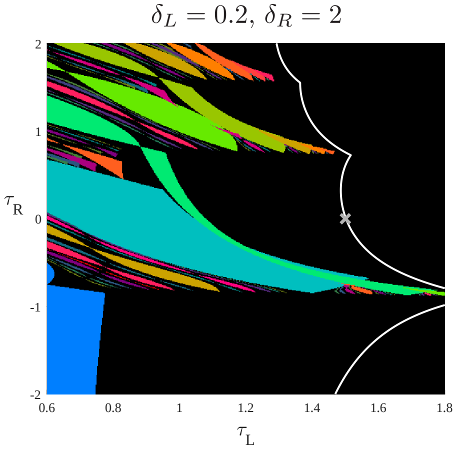


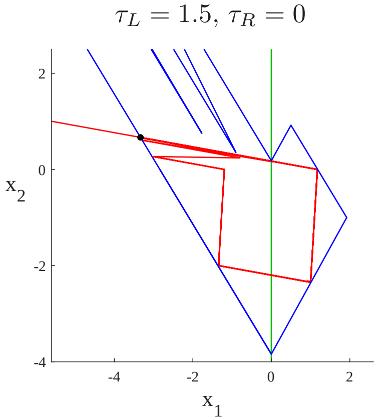
Homoclinic corners



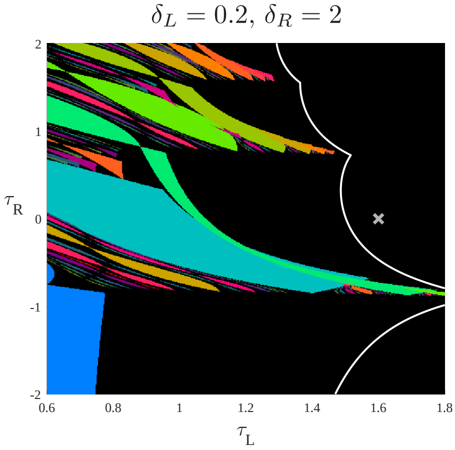


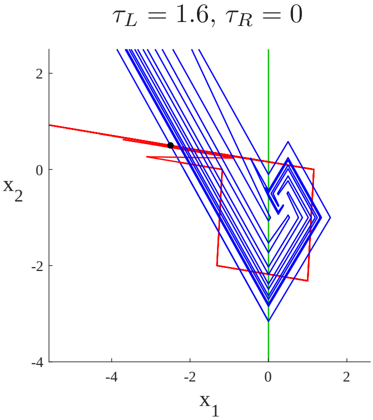
Homoclinic corners





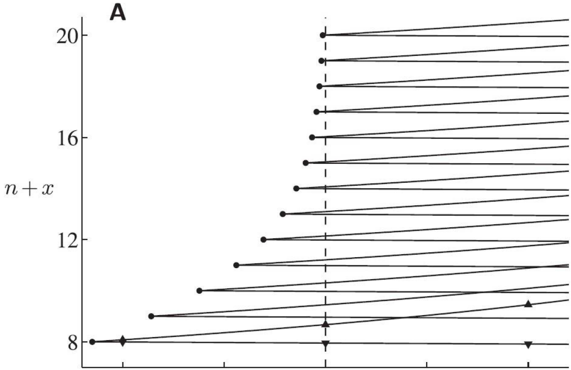
Homoclinic corners





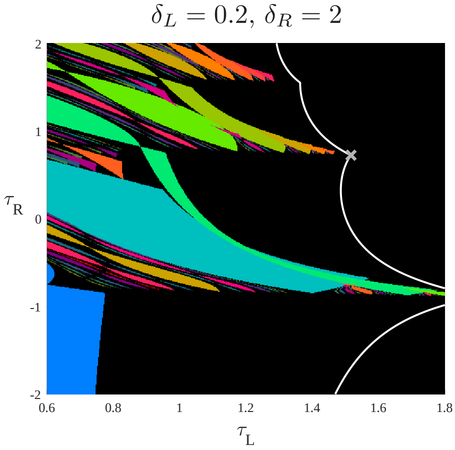
- Near a homoclinic tangency (smooth), typically get stable periodic solutions on little intervals.

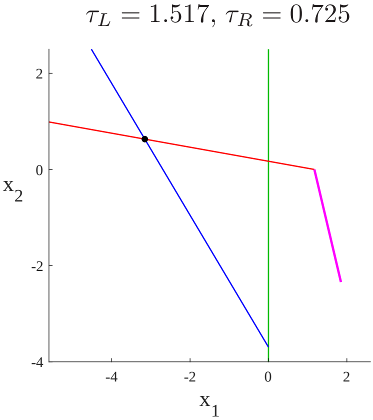
- Near a homoclinic corner (piecewise-smooth), typically get no stable periodic solutions.



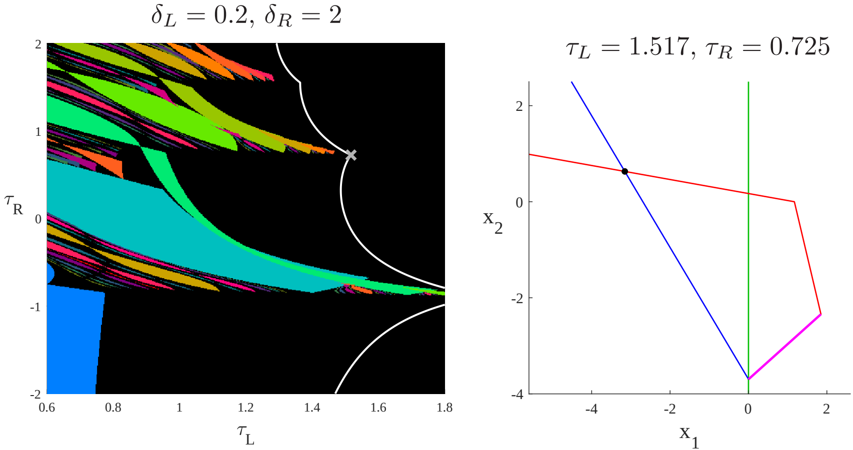
  - S., *Chaos*, 2016

Homoclinic corners

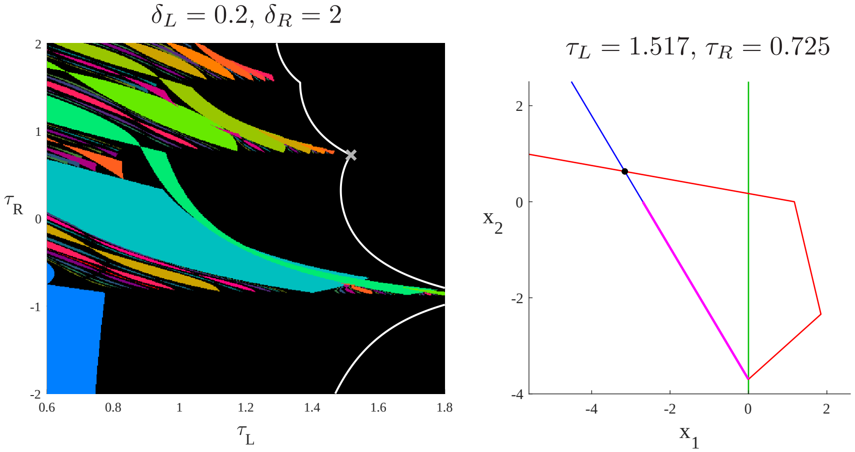




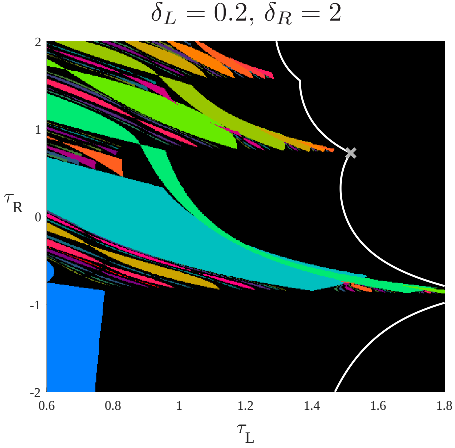
Homoclinic corners

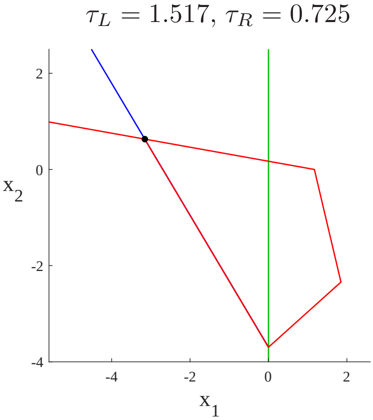


Homoclinic corners

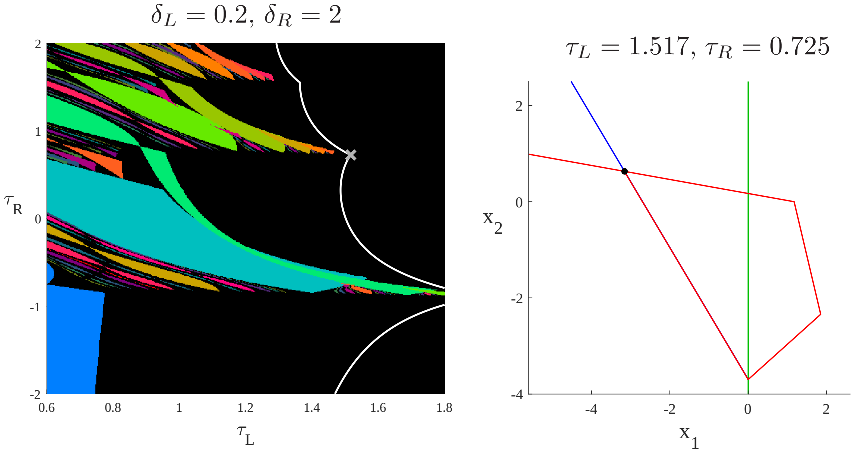


Homoclinic corners

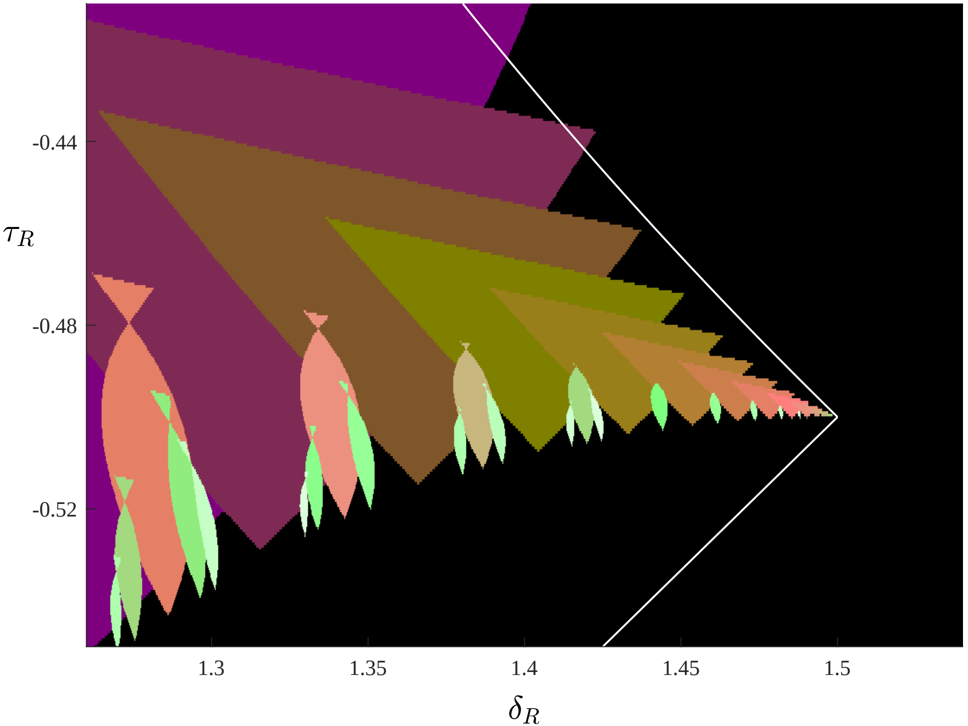




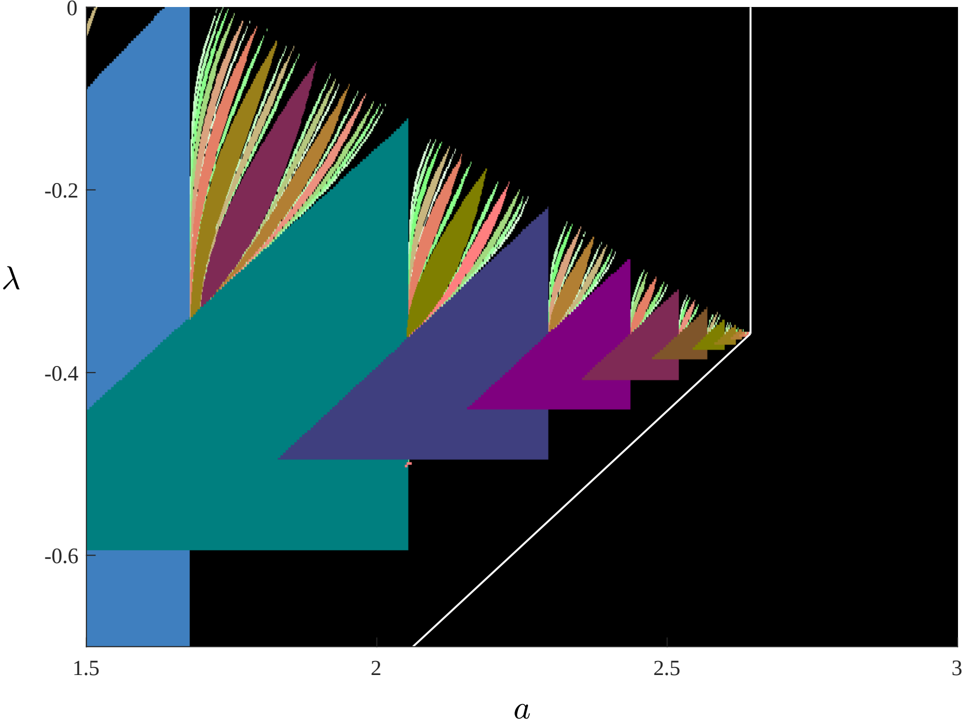
Homoclinic corners



- Such subsumed homoclinic connections are codimension-two.



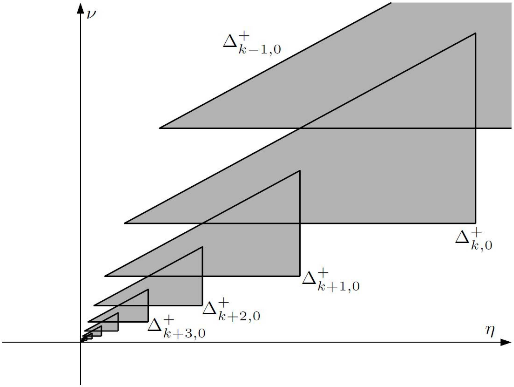
-0.44
-0.48
-0.52
τ_R
1.3
1.35
1.4
1.45
1.5
δ_R


0
-0.2
-0.4
-0.6
λ
1.5
2
2.5
3
a

- If the eigenvalues associated with the saddle satisfy

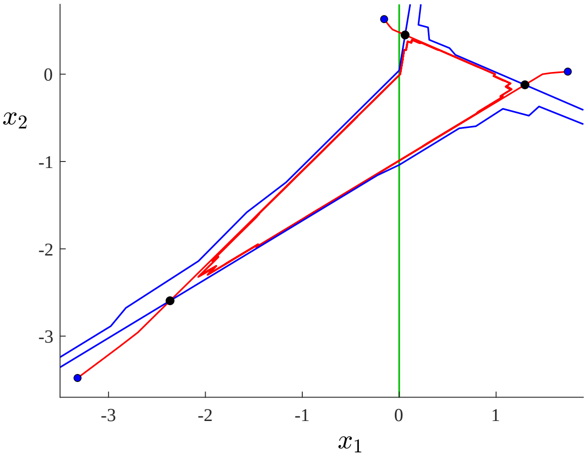
$$0 < \lambda < 1 < \sigma < \frac{1}{\lambda},$$

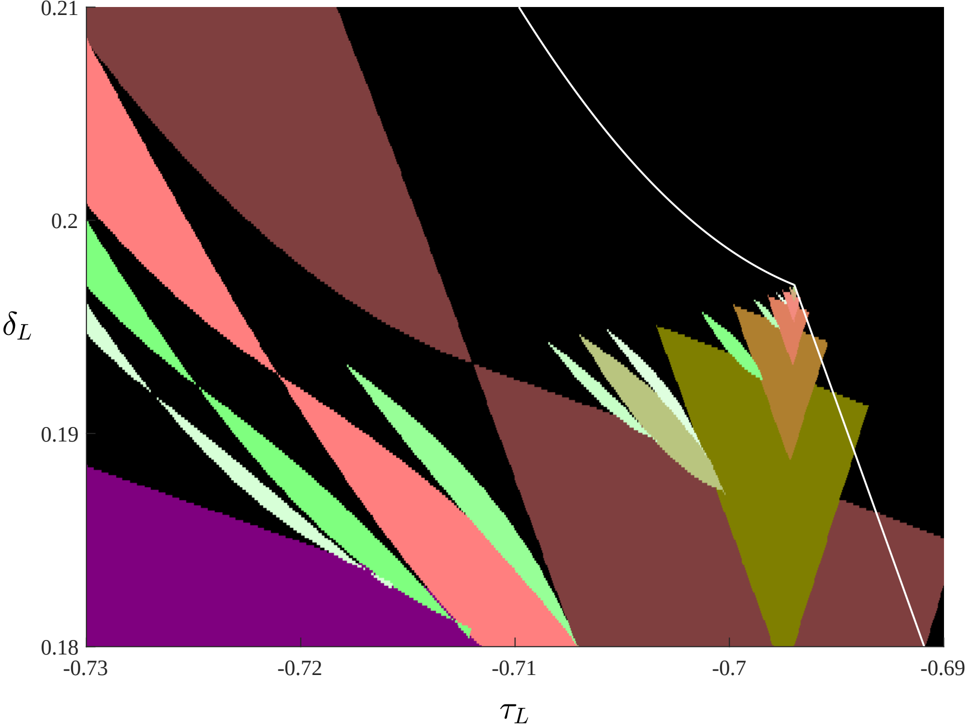
then stable periodic solutions exist in an infinite sequence of roughly triangular regions.



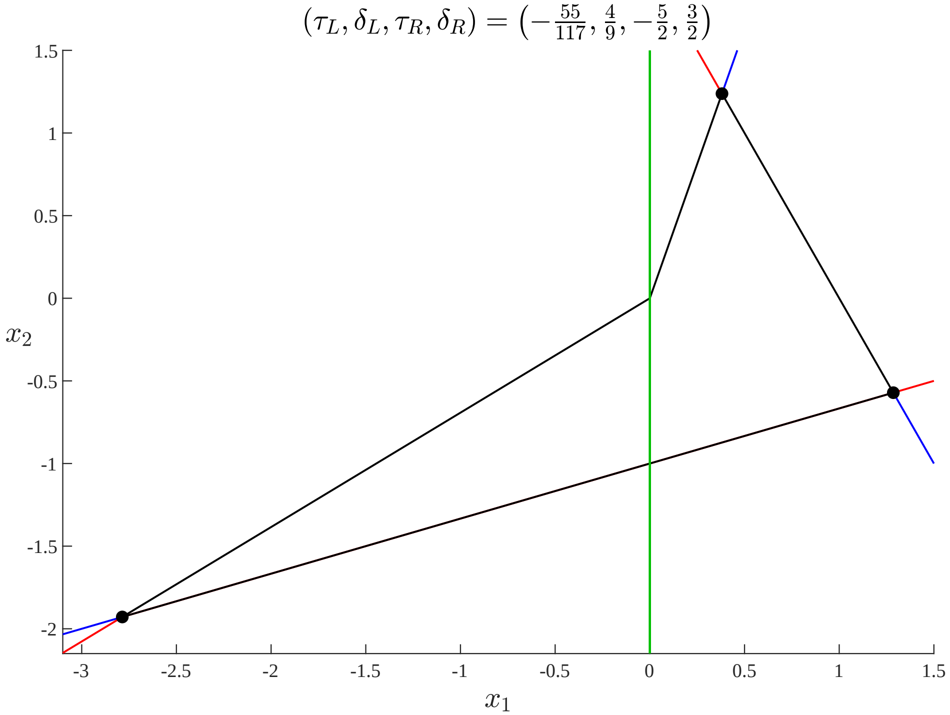
  - S., *IJBC*, 2020

- The same behaviour occurs if the saddle is a periodic solution.

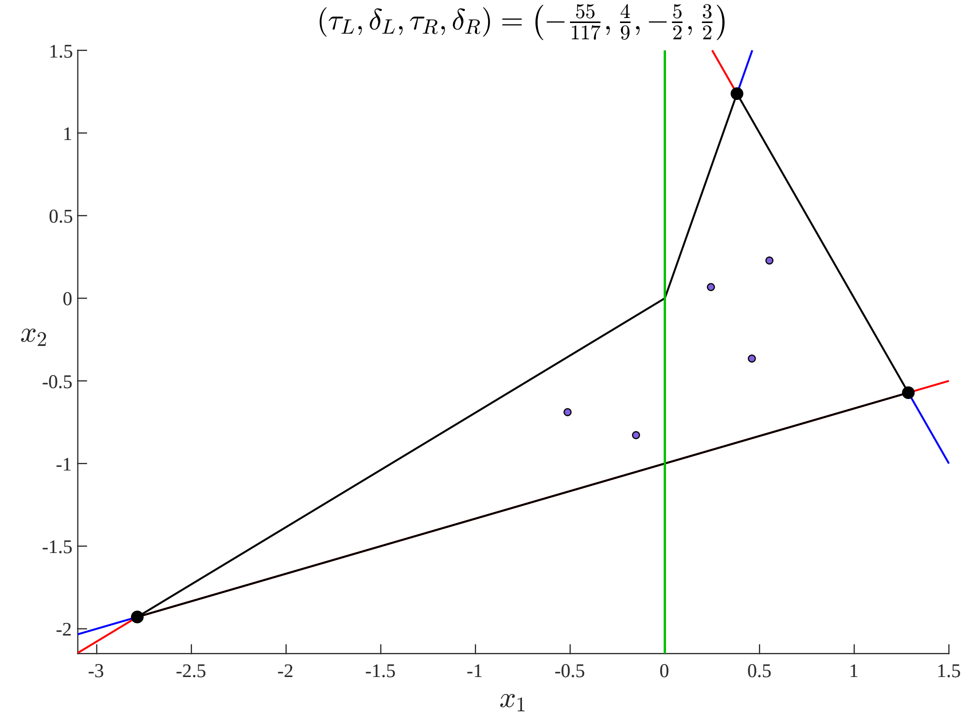


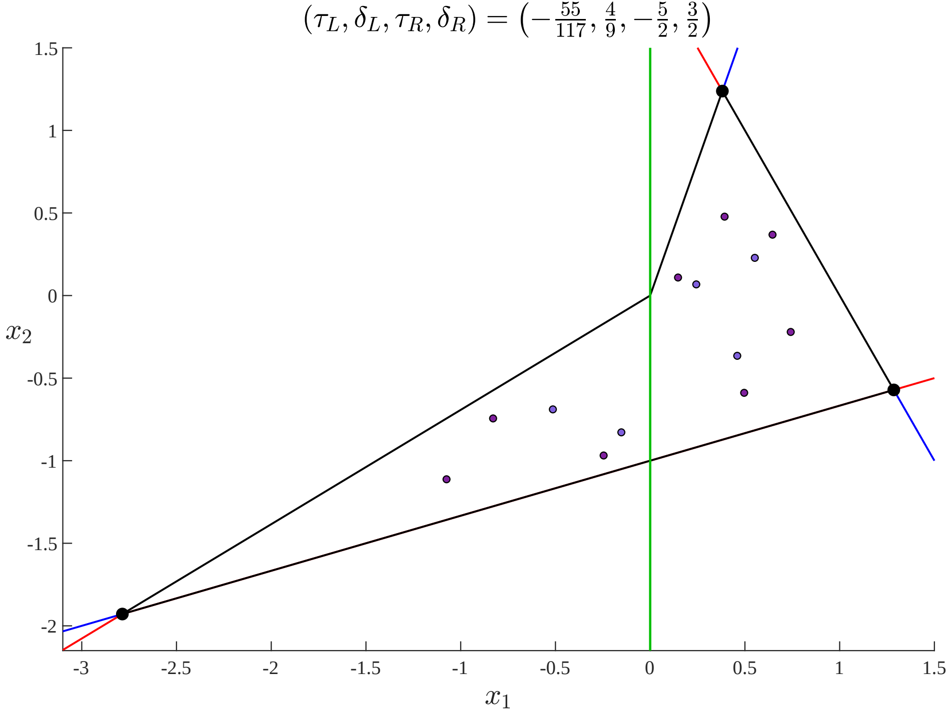

0.21
0.2
0.19
0.18
δ_L
-0.73
-0.72
-0.71
-0.7
-0.69
τ_L

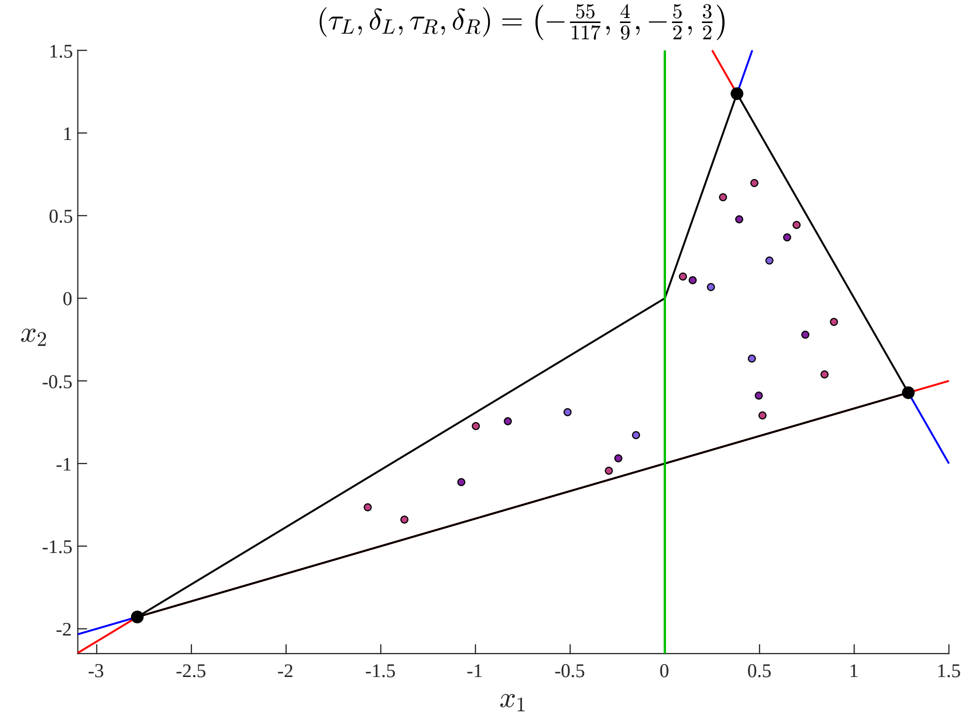
- A subsumed homoclinic connection with $\lambda\sigma = 1$ is codimension-three.
- Here there may exist infinitely many stable periodic solutions.



$(\tau_L, \delta_L, \tau_R, \delta_R) = \left(-\frac{55}{117}, \frac{4}{9}, -\frac{5}{2}, \frac{3}{2}\right)$
x_2
x_1
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5


$(\tau_L, \delta_L, \tau_R, \delta_R) = \left(-\frac{55}{117}, \frac{4}{9}, -\frac{5}{2}, \frac{3}{2}\right)$
x_1
x_2
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5


$(\tau_L, \delta_L, \tau_R, \delta_R) = \left(-\frac{55}{117}, \frac{4}{9}, -\frac{5}{2}, \frac{3}{2}\right)$
x_1
x_2
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5



$(\tau_L, \delta_L, \tau_R, \delta_R) = \left(-\frac{55}{117}, \frac{4}{9}, -\frac{5}{2}, \frac{3}{2}\right)$
x_2
x_1
1.5
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0.5
0
-0.5
-1
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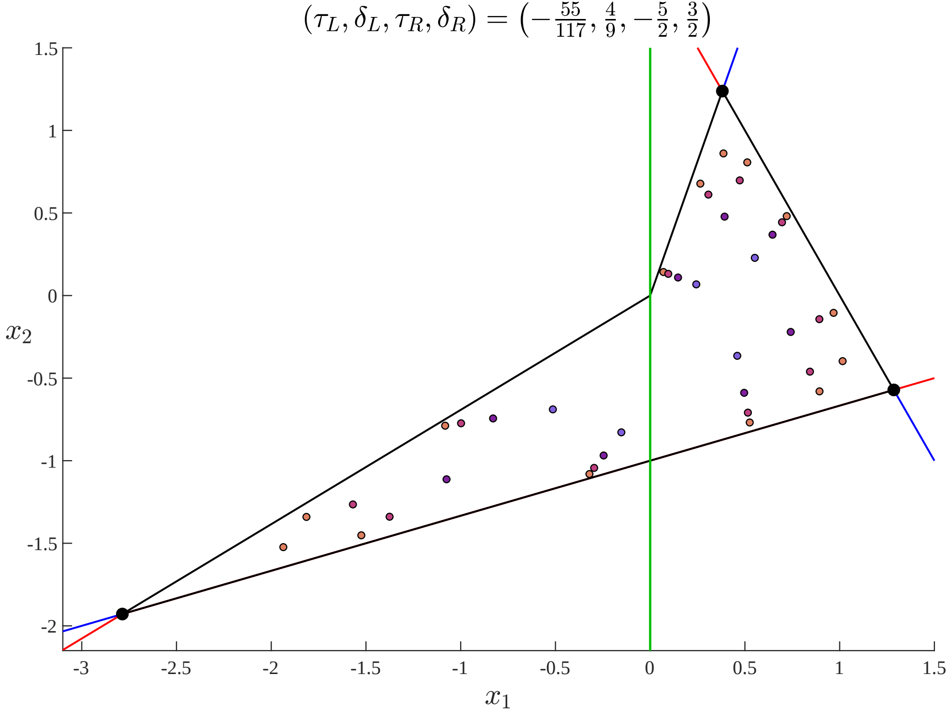
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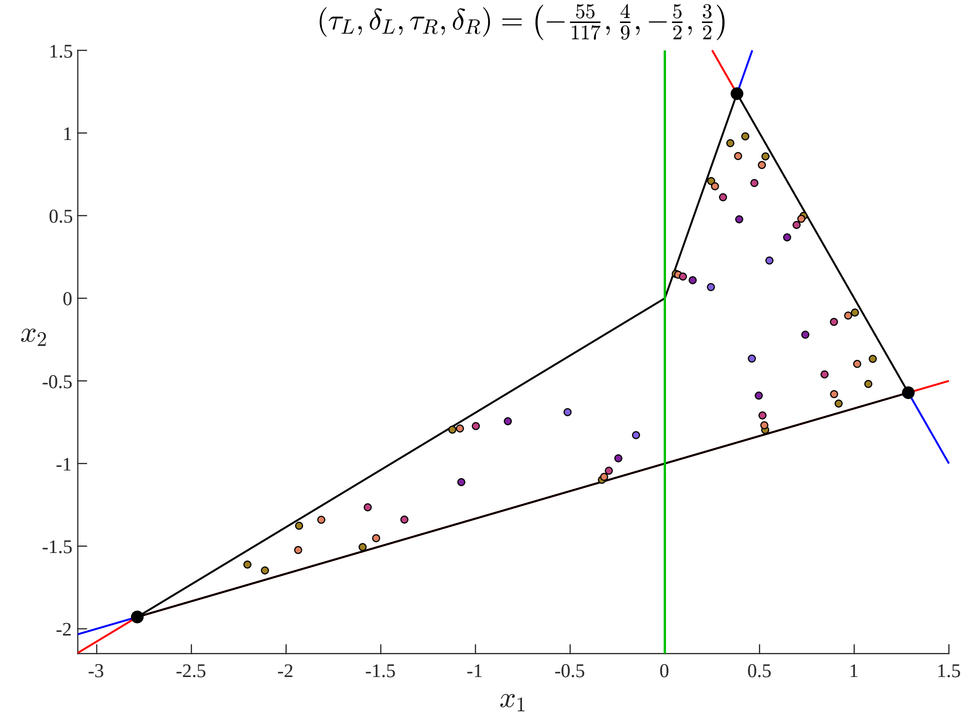
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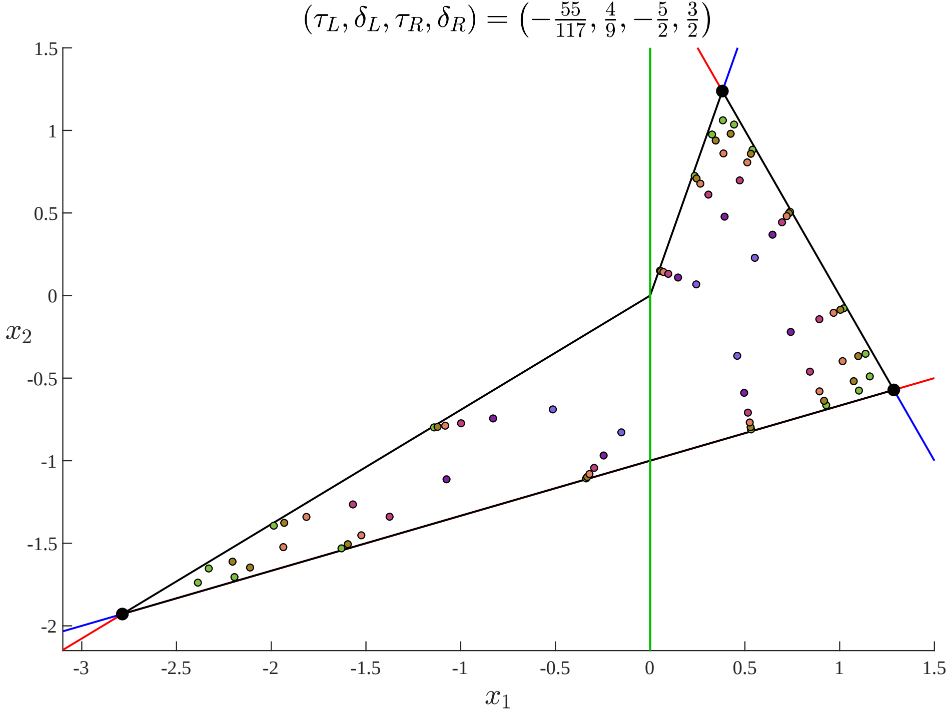
x_2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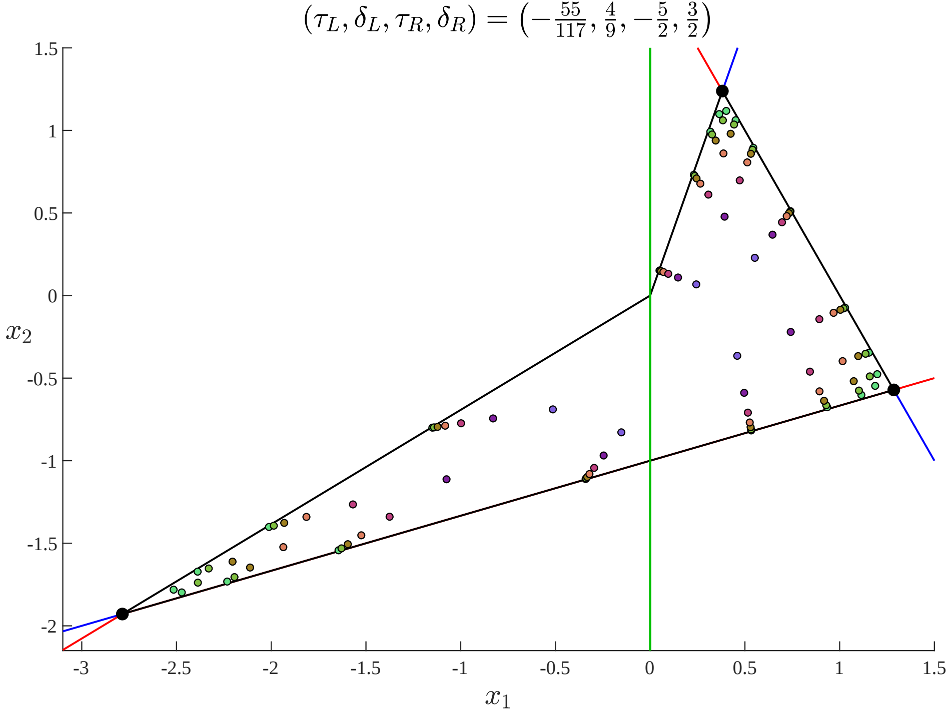

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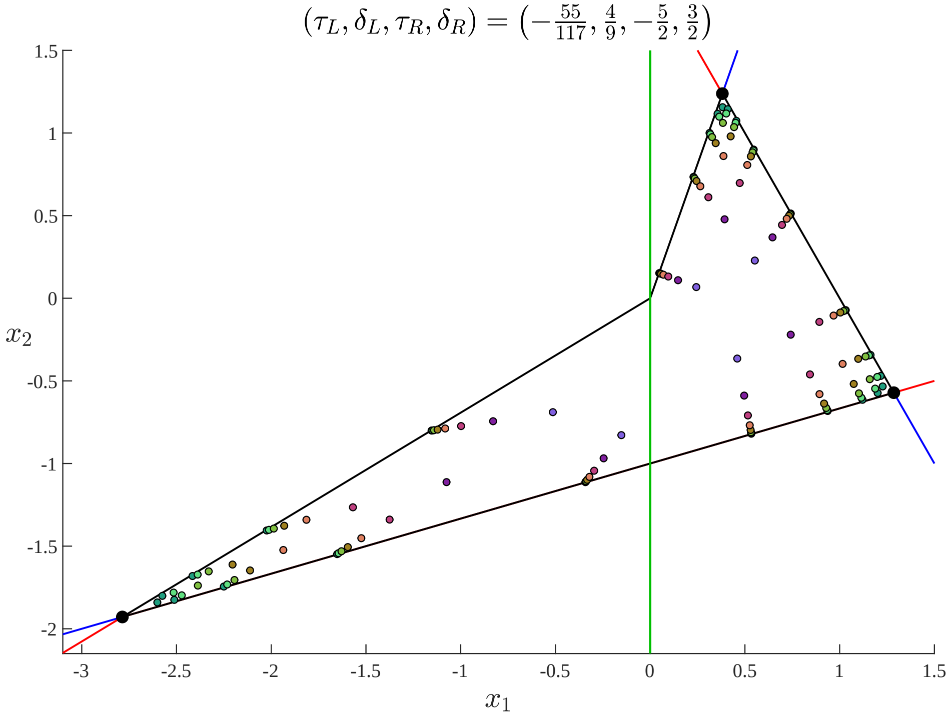

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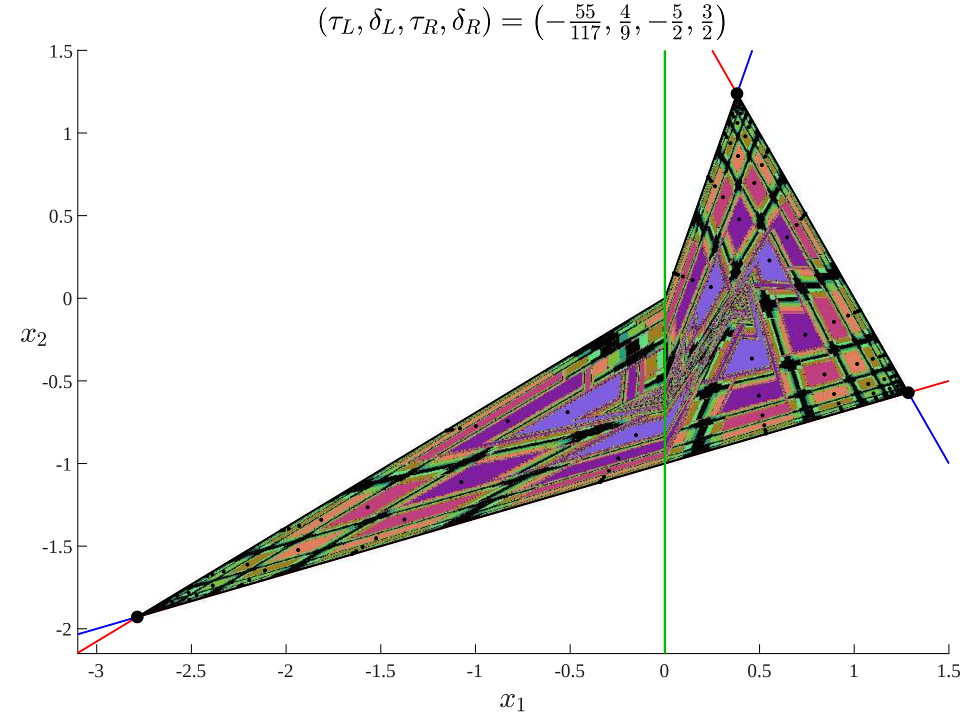

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-1.5
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-2
-1.5
-1
-0.5
0
0.5
1
1.5

- Let $\mathcal{X} \in \{L, R\}^n$ and $\mathcal{Y} \in \{L, R\}^p$ with

$$\mathcal{X}\mathcal{Y} = (\mathcal{Y}\mathcal{X})^{\overline{0}\,\overline{\alpha}}$$

  for some $\alpha \neq 0$.
- Suppose

$$M_{\mathcal{X}} = A_{\mathcal{X}_{n-1}} \cdots A_{\mathcal{X}_0}$$

  has multiplicity-one eigenvalues $0 < \lambda_2 < 1 < \lambda_1$ and all other eigenvalues have modulus less than λ_2.
- For $j = 1, 2$, let ω_j^{T} and ζ_j be left and right eigenvectors of $M_{\mathcal{X}}$ corresponding to λ_j with $\omega_j^{\mathsf{T}} \zeta_j = 1$, and let

$$C = \begin{bmatrix} \omega_1^{\mathsf{T}} \\ \omega_2^{\mathsf{T}} \end{bmatrix} M_{\mathcal{Y}} \begin{bmatrix} \zeta_1 & \zeta_2 \end{bmatrix}.$$

- Suppose $e_1^{\mathsf{T}} \zeta_1 \neq 0$.

Theorem (S & Tuffley, *IJBC*, 2017)

Suppose

i) $\lambda_1 \lambda_2 = 1$ *and* $\lambda_2 < \det(C) < 1$,

ii) *the* $\mathcal{X}$*-cycle has no points on* Σ *(the switching manifold),*

iii) *there exists an* $\mathcal{X}^\infty \mathcal{Y} \mathcal{X}^\infty$*-orbit* $\{p_i\}$ *homoclinic to the* $\mathcal{X}$*-cycle and with* $p_0 = E^u\left(X_0^{\mathcal{X}}\right) \cap \Sigma$ *and* $p_\alpha \in \Sigma$*, and*

iv) *there is no* $i \geq 0$ *for which* $p_i \in \Sigma$ *and* $p_{i+n} \in \Sigma$*.*

Then there exists $k_{\min}$ *such that* f *has an attracting* $\mathcal{X}^k \mathcal{Y}$*-cycle for all* $k \geq k_{\min}$*.*

THEOREM (S & TUFFLEY, *IJBC*, 2017)

Suppose

i) $\det(C) \notin \{-1, 0\}$,

ii) $f_{\mathcal{Y}^{\overline{0}}}\big(E^u\big(X_0^{\mathcal{X}}\big)\big) \not\subset E^s\big(X_0^{\mathcal{X}}\big)$, *and*

iii) *there exists* $k_{\min}$ *such that* f *has an attracting* $\mathcal{X}^k\mathcal{Y}$*-cycle for all* $k \geq k_{\min}$.

Then there exists an $\mathcal{X}^\infty\mathcal{Y}\mathcal{X}^\infty$*-orbit* $\{p_i\}$ *(possibly virtual) homoclinic to the* $\mathcal{X}$*-cycle and with* $p_0 = E^u\big(X_0^{\mathcal{X}}\big) \cap \Sigma$ *and* $p_\alpha \in \Sigma$*, and* $\lambda_1\lambda_2 = 1$.

Occurrence in an ODE system

$$\begin{bmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \\ \frac{dw}{dt} \end{bmatrix} = \begin{cases} \begin{bmatrix} v \\ w \\ -\alpha_1(u+1) - \alpha_2 v - \alpha_3 w + \gamma\cos(t) \end{bmatrix}, & u < 0 \\ \begin{bmatrix} -1 \\ \beta_1 \\ \beta_2 \end{bmatrix}, & u > 0 \end{cases}$$

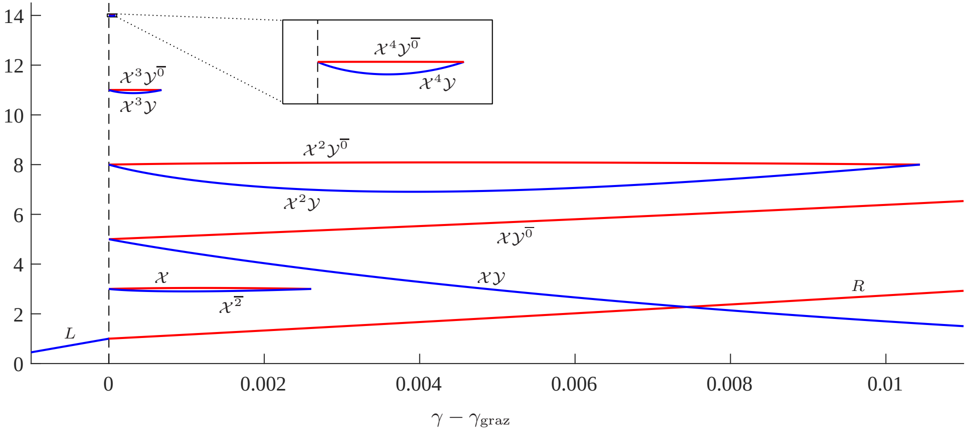
$$\alpha_1 \approx 0.0302445699$$

$$\alpha_2 \approx 0.1667559781$$

$$\alpha_3 \approx 0.4009520660$$

$$\beta_1 \approx -0.3783802961$$

$$\beta_2 \approx -0.5981255840$$



14
12
10
8
6
4
2
0
0
0.002
0.004
0.006
0.008
0.01
$\gamma - \gamma_{\mathrm{graz}}$
L
R
$\mathcal{X}$
$\mathcal{X}^{\overline{2}}$
$\mathcal{X}\mathcal{Y}$
$\mathcal{X}\mathcal{Y}^{\overline{0}}$
$\mathcal{X}^2\mathcal{Y}$
$\mathcal{X}^2\mathcal{Y}^{\overline{0}}$
$\mathcal{X}^3\mathcal{Y}$
$\mathcal{X}^3\mathcal{Y}^{\overline{0}}$
$\mathcal{X}^4\mathcal{Y}$
$\mathcal{X}^4\mathcal{Y}^{\overline{0}}$

Outline

1) A sausage-string structure for periodicity regions.
2) Homoclinic corners and infinitely many attractors.
3) Stability of boundary fixed points.

- Set $\mu = 0$:

$$x \mapsto \begin{cases} A_L x, & x_1 \leq 0, \\ A_R x, & x_1 \geq 0. \end{cases} \tag{$\bigstar$}$$

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- The origin is a fixed point on the switching manifold.

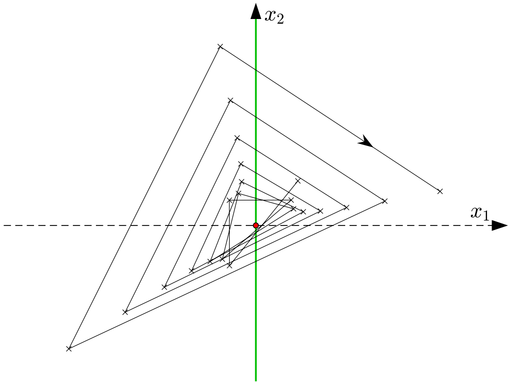
- Set $\mu = 0$:

$$x \mapsto \begin{cases} A_L x, & x_1 \leq 0, \\ A_R x, & x_1 \geq 0. \end{cases} \qquad (\bigstar)$$

- The origin is a fixed point on the switching manifold.
- PROBLEM: for what A_L and A_R is it stable?

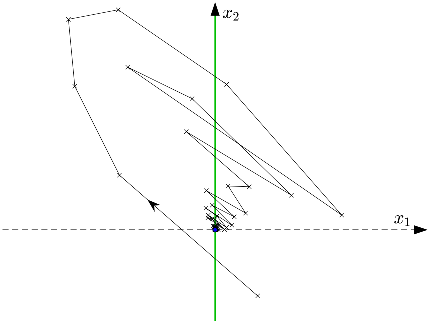
- If $n \geq 2$, even if all eigenvalues of A_L and A_R have modulus less than 1, $\mathbf{0}$ can be an unstable fixed point of ($\bigstar$).
  - Hassouneh, Abed, and Nusse, *Phys. Rev. Lett.*, 2004
  - Do and Baek, *Commun. Pure Appl. Anal.*, 2006

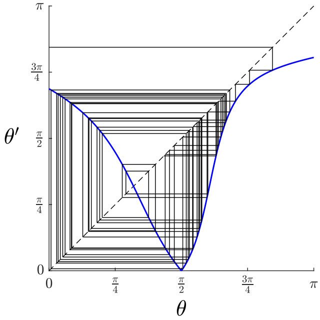
  - Example:
    A_L has eigenvalues: $0.95\,\mathrm{e}^{\pm 1.8\mathrm{i}}$
    A_R has eigenvalues: $0.95\,\mathrm{e}^{\pm 2.6\mathrm{i}}$



$$x \mapsto \begin{cases} A_L x, & x_1 \le 0, \\ A_R x, & x_1 \ge 0. \end{cases} \qquad (\bigstar)$$

- If $n \ge 2$ and the map is non-invertible, orbits can converge to $\mathbf{0}$ in a chaotic fashion:





$$x \mapsto \begin{cases} A_L x, & x_1 \leq 0, \\ A_R x, & x_1 \geq 0. \end{cases} \tag{$\bigstar$}$$

$$x \mapsto \begin{cases} A_L x + o(x), & x_1 \leq 0, \\ A_R x + o(x), & x_1 \geq 0. \end{cases} \tag{$\bigstar\,\bigstar$}$$

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THEOREM (S., *J. Dyn. Diff. Eq.*, 2020)

If $\mathbf{0}$ *is asymptotically stable for* ($\bigstar$)
then it is also asymptotically stable for ($\bigstar\bigstar$).

$$x \mapsto \begin{cases} A_L x, & x_1 \leq 0, \\ A_R x, & x_1 \geq 0. \end{cases} \tag{$\bigstar$}$$

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Theorem (S., *J. Dyn. Diff. Eq.*, 2020)

If $\mathbf{0}$ *is asymptotically stable for* ($\bigstar$)
then it is also asymptotically stable for ($\bigstar\bigstar$).

Conjecture

If $\mathbf{0}$ *is unstable for* ($\bigstar$)
then it is also unstable for ($\bigstar\bigstar$).

- Stability can be determined computationally.
  - Gardini, *Nonlin. Anal.*, 1992
  - Athanasopoulos and Lazar, *IFAC Proceedings*, 2014

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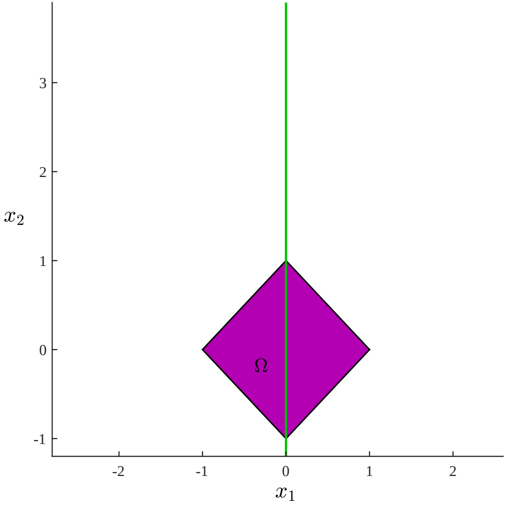
Theorem (S., *NZJM*, 2020)

Let f be a map of the form $(\bigstar)$. Let $\Omega \subset \mathbb{R}^n$ be compact with $\mathbf{0} \in \mathrm{int}(\Omega)$. The following are equivalent.

1) $\mathbf{0}$ *is an asymptotically stable fixed point of f.*
2) *There exists $m \geq 1$ such that $f^m(\Omega) \subset \mathrm{int}(\Omega)$.*
3) *There exists $p \geq 1$ such that $f^p(\Omega) \subset \mathrm{int}\left(\bigcup_{i=0}^{p-1} f^i(\Omega)\right)$.*

- Let Ω be a polytope. Then $f^i(\Omega)$ is a polytope for all $i \geq 0$ and each $\bigcup_{i=0}^{p-1} f^i(\Omega)$ can be encoded with finitely many data points.
    - Here stability is verified with $p = 12$:



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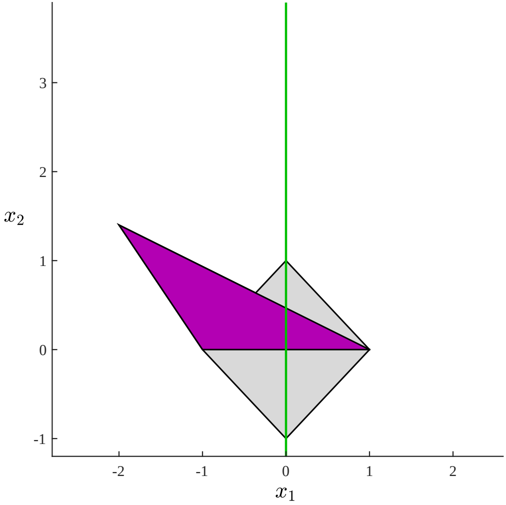

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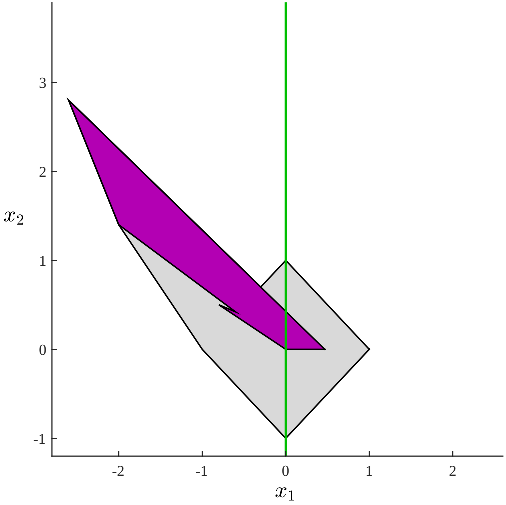

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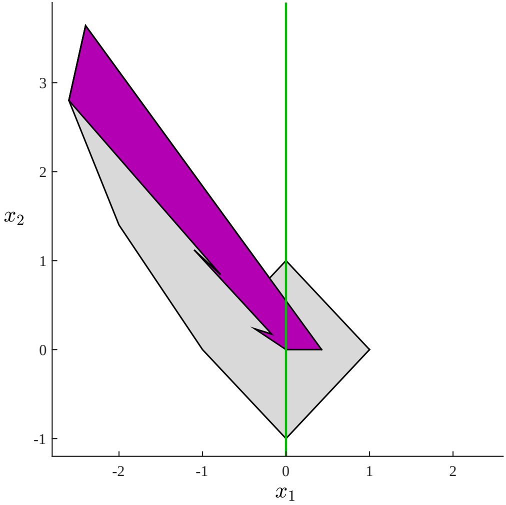

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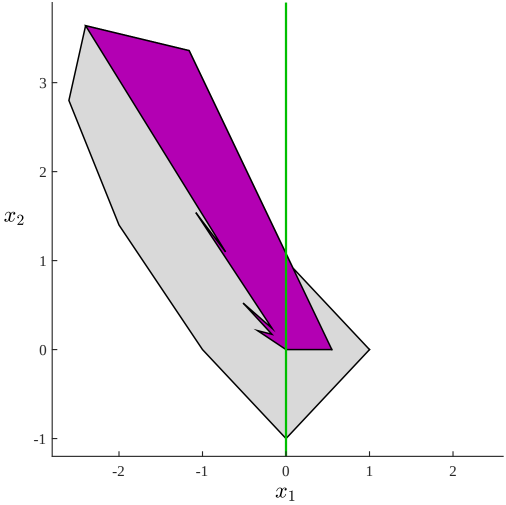

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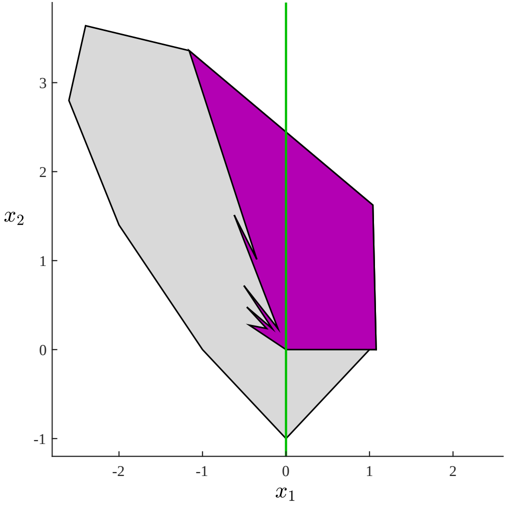

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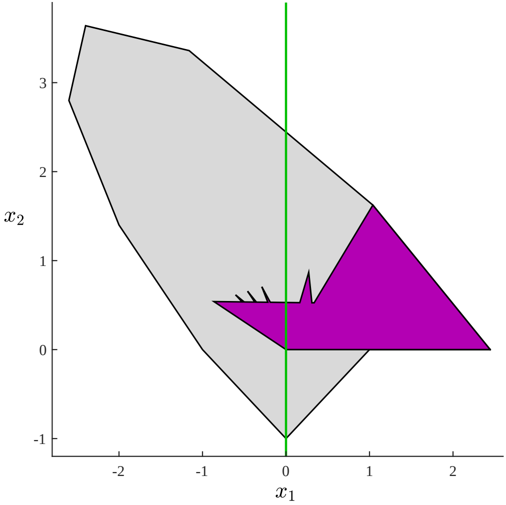

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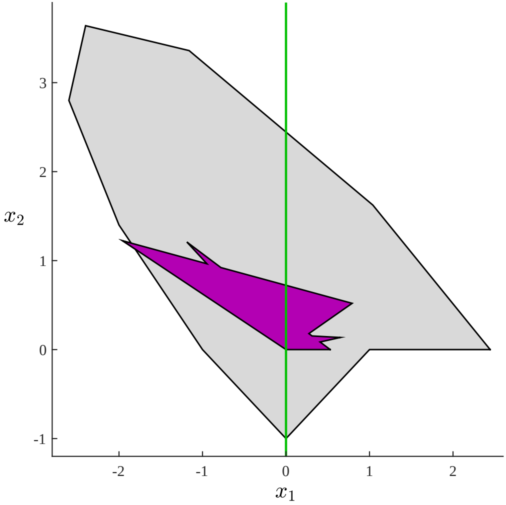

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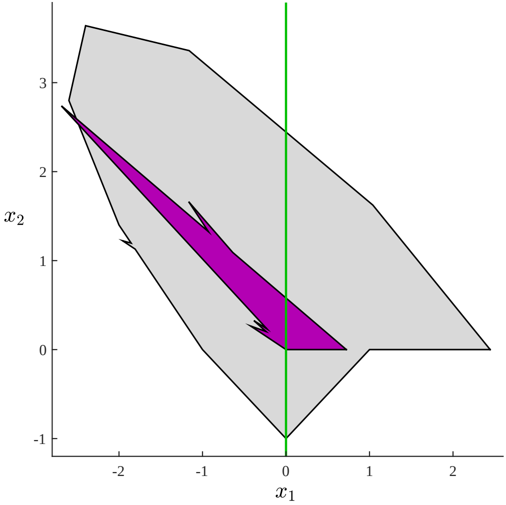

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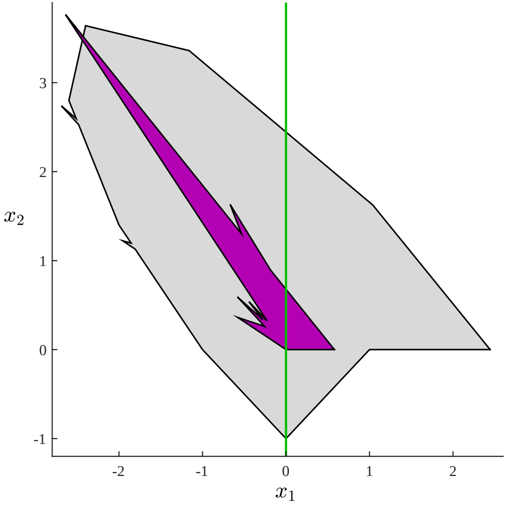

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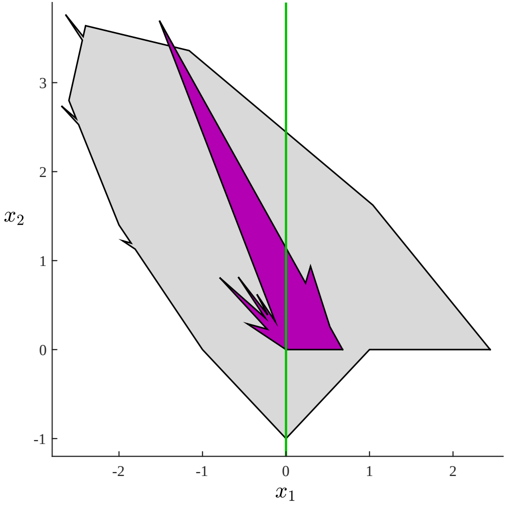

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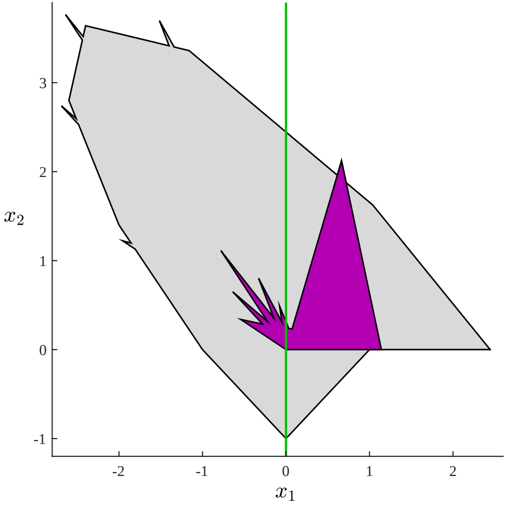

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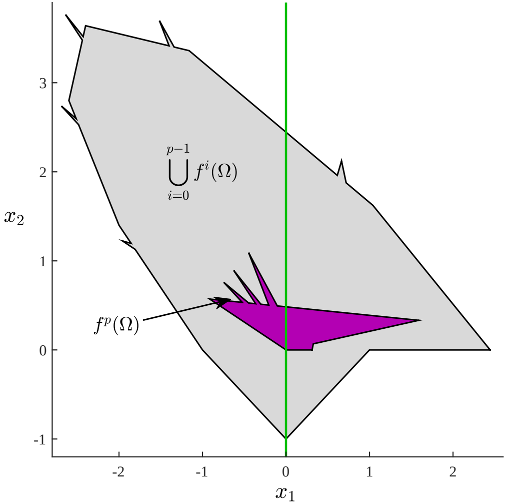
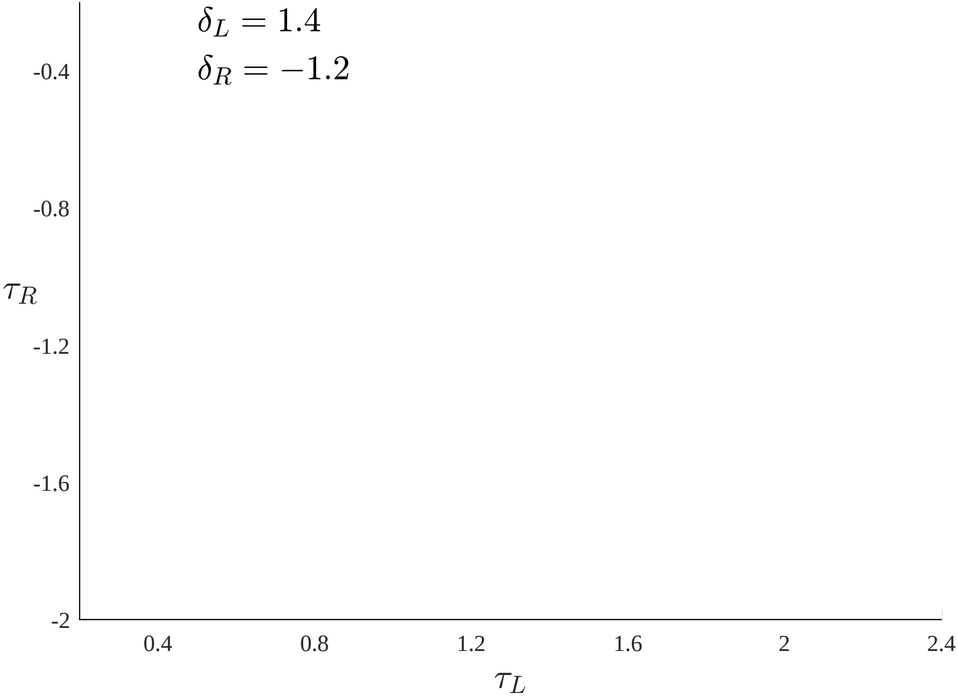
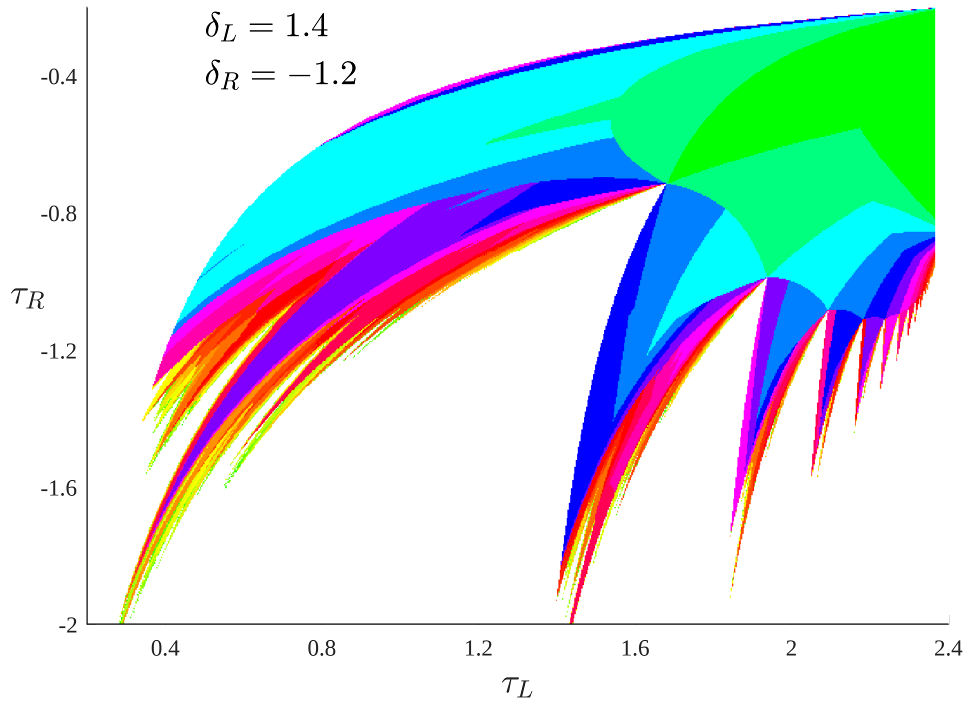


- With $n = 2$ the map has four parameters:

$$x \mapsto \begin{cases} \begin{bmatrix} \tau_L & 1 \\ -\delta_L & 0 \end{bmatrix} x, & x_1 \leq 0, \\ \begin{bmatrix} \tau_R & 1 \\ -\delta_R & 0 \end{bmatrix} x, & x_1 \geq 0. \end{cases}$$



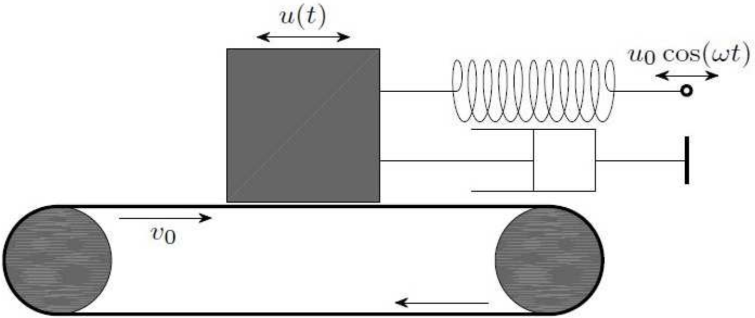
$\delta_L = 1.4$
$\delta_R = -1.2$
-0.4
-0.8
τ_R
-1.2
-1.6
-2
0.4
0.8
1.2
1.6
2
2.4
τ_L


$\delta_L = 1.4$
$\delta_R = -1.2$
-0.4
-0.8
-1.2
-1.6
-2
τ_R
0.4
0.8
1.2
1.6
2
2.4
τ_L

- If $\mathbf{0}$ is asymptotically stable with $\mu = 0$, then there exists a local attractor with $\mu \neq 0$.

Stick-slip friction oscillator

$$\ddot{u} + 2\zeta_r \dot{u} + u = u_0 \cos(\omega t) - F_s \mathrm{sgn}\left(\dot{u} - v_0\right) - \kappa v_0$$



- Grazing-sliding bifurcations described by

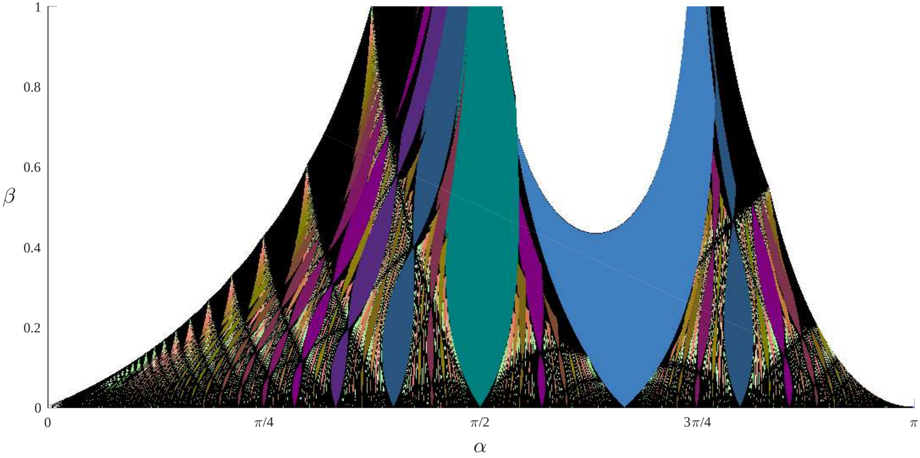
$$f(x,y) = \begin{cases} \begin{bmatrix} 2e^{\beta}\cos(\alpha) & 1 \\ -e^{2\beta} & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \mu \\ 0 \end{bmatrix}, & x \le 0, \\ \begin{bmatrix} e^{\beta}\cos(\alpha) & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \mu \\ 0 \end{bmatrix}, & x \ge 0, \end{cases}$$

  where

$$\alpha = \frac{2\pi\sqrt{1-\zeta_r^2}}{\omega}, \qquad \beta = -\frac{2\pi\zeta_r}{\omega}.$$

  - Szalai and Osinga, *Chaos*, 2008.

Dynamics for $\mu < 0$



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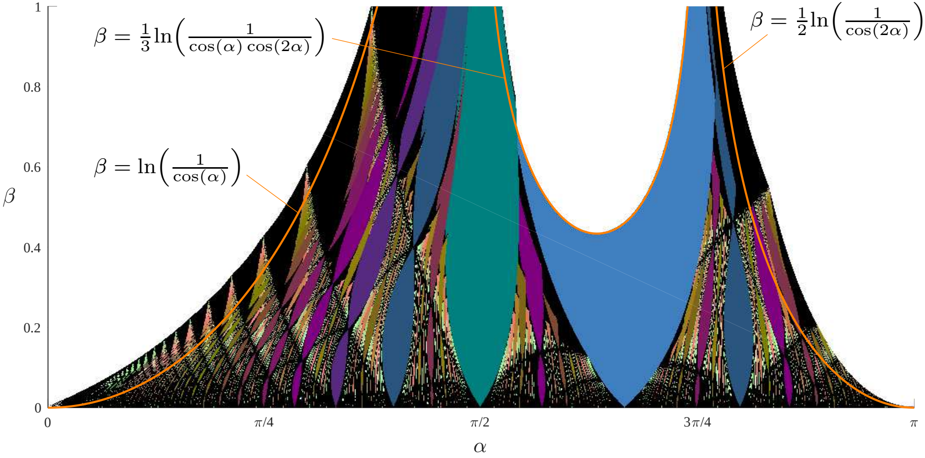

Summary

- Rotational periodicity regions have a sausage-string structure.
  - What structures do non-rotational periodicity regions have?
- Near a subsumed homoclinic connection get roughly triangular periodicity regions.
  - Between these how are other periodicity regions organised?
- Asymptotic stability of a fixed point on a switching manifold is robust to higher order terms.
  - What about instability?
- Computational methods exist for determining stability.
  - How to extend these to ODEs (e.g. Filippov systems)?