

UNFOLDING GLOBALLY RESONANT HOMOCLINIC TANGENCIES

SISHU SHANKAR MUNI*

ROBERT I. McLACHLAN AND DAVID J. W. SIMPSON

School of Fundamental Sciences, Massey University
 Palmerston North
 New Zealand

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ABSTRACT. Global resonance is a mechanism by which a homoclinic tangency of a smooth map can have infinitely many asymptotically stable, single-round periodic solutions. To understand the bifurcation structure one would expect to see near such a tangency, in this paper we study one-parameter perturbations of typical globally resonant homoclinic tangencies. We assume the tangencies are formed by the stable and unstable manifolds of saddle fixed points of two-dimensional maps. We show the perturbations display two infinite sequences of bifurcations, one saddle-node the other period-doubling, between which single-round periodic solutions are asymptotically stable. The distance of the bifurcation values from global resonance generically scales like $|\lambda|^{2k}$, as $k \rightarrow \infty$, where $-1 < \lambda < 1$ is the stable eigenvalue associated with the fixed point. If the perturbation is taken tangent to the surface of codimension-one homoclinic tangencies, the scaling is instead like $\frac{|\lambda|^k}{k}$. We also show slower scaling laws are possible if the perturbation admits further degeneracies.

1. Introduction. Homoclinic tangencies are perhaps the simplest mechanism in nonlinear dynamical systems for the loss of hyperbolicity and the creation of chaotic dynamics [12]. They occur most simply for saddle fixed points of two-dimensional maps. A tangential intersection between the stable and unstable manifolds of a fixed point is a homoclinic tangency, see Fig 1. This intersection is one point of an orbit that is homoclinic to the fixed point. This is a codimension-one phenomenon, meaning it can be equated to a single scalar condition. At a homoclinic tangency the map typically has infinitely many periodic solutions. Some of these are *single-round*, roughly meaning that they shadow the homoclinic orbit once before repeating, Fig. 1. At a generic homoclinic tangency all single-round periodic solutions of sufficiently large period are unstable [3, 4].

In our previous work [10] we determined what degeneracies are needed for infinitely many single-round periodic solutions to be asymptotically stable for smooth maps on $\mathbb{R}^2$. We found that the eigenvalues associated with the saddle fixed point,

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* Corresponding author: Sishu Shankar Muni.

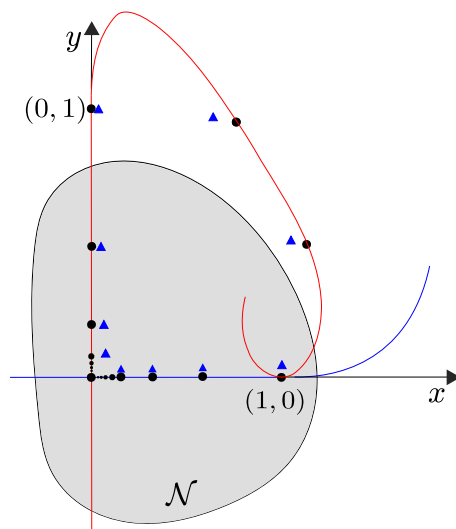


FIGURE 1. A homoclinic tangency for a saddle fixed point of a two-dimensional map. In this illustration the eigenvalues associated with the fixed points are positive, i.e. $0 < \lambda < 1$ and $\sigma > 1$. A coordinate change has been applied so that in the region $\mathcal{N}$ (shaded) the coordinate axes coincide with the stable and unstable manifolds. The homoclinic orbit Γ_{HC} is shown with black dots. A typical single-round periodic solution is shown with blue triangles.

call them λ and σ satisfying $0 < |\lambda| < 1 < |\sigma|$, need to multiply to 1 or -1 . If $\lambda\sigma = -1$ (so the map is orientation-reversing, at least locally), then the phenomenon is codimension-three. It involves a ‘global resonance’ condition on the reinjection mechanism for taking iterates back to a neighbourhood of the fixed point.

If a family of maps f_μ , with $\mu \in \mathbb{R}^n$, has this phenomenon at some point μ^* in parameter space, then for any positive $j \in \mathbb{Z}$ there exists an open set containing μ^* in which j asymptotically stable, single-round periodic solutions coexist. Due to the high codimension, a precise description of the shape of these sets (for large j) is beyond the scope of this paper. The approach we take here is to consider one-parameter families that perturb from a globally resonant homoclinic tangency. Some information about the size and shapes of the sets can then be inferred from our results. Globally resonant homoclinic tangencies are hubs for extreme multi-stability. They should occur generically in some families of maps with three or more parameters, such as the generalised Hénon map [8, 9], but to our knowledge they have yet to have been identified.

We find that as the value of μ is varied from μ^* , there occurs an infinite sequence of either saddle-node or period-doubling bifurcations that destroy the periodic solutions or make them unstable. Generically these sequences converge exponentially to μ^* with the distance (in parameter space) to the bifurcation asymptotically proportional to $|\lambda|^{2k}$, where the periodic solutions have period $k + m$, for some fixed $m \geq 1$. If we move away from μ^* without a linear change to the codimension-one condition of a homoclinic tangency, the bifurcation values instead generically scale like $\frac{|\lambda|^k}{k}$. If the perturbation suffers further degeneracies, the scaling can be

slower. Specifically we observe $|\lambda|^k$ and $\frac{1}{k}$ for an abstract example that we believe is representative of how the bifurcations scale in general.

Similar results have been obtained for more restrictive classes of maps. For area-preserving families the phenomenon is codimension-two and there exist infinitely many elliptic, single-round periodic solutions [6]. As shown in [1, 5] the periodic solutions are destroyed or lose stability in bifurcations that scale like $|\lambda|^{2k}$, matching our result. For piecewise-linear families the phenomenon is codimension-three [15, 16]. In this setting the bifurcation values instead scale like $|\lambda|^k$ [14], see also [2].

The phenomenon is reminiscent of a celebrated result of Newhouse [11]. Newhouse showed that for a generic (codimension-one) homoclinic tangency with $|\lambda\sigma| < 1$, infinitely many asymptotically stable periodic solutions coexist for a dense set of parameter values near that of the tangency. In this scenario the periodic solutions are *multi-round* (shadow the homoclinic orbit more than once before repeating).

The remainder of the paper is organised as follows. In §2 we introduce two maps, T_0 and T_1 , to describe the dynamics near the homoclinic orbit. The map T_0 is a local map describing the saddle fixed point, while T_1 represents the reinjection mechanism. The single-round periodic solutions correspond to fixed points of $T_0^k \circ T_1$. In §3 we summarise the necessary and sufficient conditions derived in [10] for the coexistence of infinitely many asymptotically stable, single-round periodic solutions. Then in §4–6 we introduce parameter-dependence and state our main result (Theorem 6.1) for the scaling properties of the saddle-node and period-doubling bifurcations. A proof of Theorem 6.1 is provided in §7. In §8 we illustrate Theorem 6.1 with a four-parameter family of C^1 maps (an abstract minimal example). We observe numerically computed bifurcation values match those predicted by Theorem 6.1, to leading order, and observe slower scaling laws in special cases. Finally §9 provides a discussion and outlook for further studies.

2. A quantitative description of the dynamics near a homoclinic connection. Let f be a C^∞ map on $\mathbb{R}^2$. Suppose the origin $(x, y) = (0, 0)$ is a saddle fixed point of f . That is, $Df(0, 0)$ has eigenvalues $\lambda, \sigma \in \mathbb{R}$ satisfying

$$0 < |\lambda| < 1 < |\sigma|. \quad (1)$$

By the results of Sternberg [13, 17] there exists a C^∞ coordinate change that transforms f to

$$T_0(x, y) = \begin{bmatrix} \lambda x(1 + \mathcal{O}(xy)) \\ \sigma y(1 + \mathcal{O}(xy)) \end{bmatrix}. \quad (2)$$

In these new coordinates let $\mathcal{N}$ be a convex neighbourhood of the origin for which

$$f(x, y) = T_0(x, y), \quad \text{for all } (x, y) \in \mathcal{N}, \quad (3)$$

see Fig. 1. If $\lambda^p \sigma^q \neq 1$ for all integers $p, q \geq 1$, then the eigenvalues are said to be *non-resonant* and the coordinate change can be chosen so that T_0 is linear. If not, then T_0 contains resonant terms that cannot be eliminated by the coordinate change. As explained in §4, if $\lambda\sigma = 1$ we can reach the form

$$T_0(x, y) = \begin{bmatrix} \lambda x(1 + a_1 xy + \mathcal{O}(x^2 y^2)) \\ \sigma y(1 + b_1 xy + \mathcal{O}(x^2 y^2)) \end{bmatrix}, \quad (4)$$

where $a_1, b_1 \in \mathbb{R}$. If $\lambda\sigma = -1$ we can obtain (4) with $a_1 = b_1 = 0$.

Now suppose there exists an orbit homoclinic to the origin, Γ_{HC} . By scaling x and y we may assume that $(1, 0)$ and $(0, 1)$ are points on Γ_{HC} and

$$(1, 0), (\lambda, 0), (0, \frac{1}{\sigma}), (0, \frac{1}{\sigma^2}) \in \mathcal{N}. \quad (5)$$

By assumption there exists $m \geq 1$ such that $f^m(0, 1) = (1, 0)$. We let $T_1 = f^m$ and expand T_1 in a Taylor series centred at $(x, y) = (0, 1)$:

$$T_1(x, y) = \begin{bmatrix} c_0 + c_1x + c_2(y-1) + \mathcal{O}((x, y-1)^2) \\ d_0 + d_1x + d_2(y-1) + d_3x^2 + d_4x(y-1) + d_5(y-1)^2 + \mathcal{O}((x, y-1)^3) \end{bmatrix}, \quad (6)$$

where $c_0 = 1$ and $d_0 = 0$. In (6) we have written explicitly the terms that will be important below.

3. Conditions for infinitely many stable single-round solutions. In this section we state the main results of our previous work [10]. First Theorem 3.1 gives necessary conditions for the existence of infinitely many stable, single-round periodic solutions. Then Theorem 3.2 gives sufficient conditions for these to exist and be asymptotically stable.

Theorem 3.1. *Suppose f satisfies (3) with (2) and (5). Suppose in (6) we have $c_0 = 1$, $d_0 = 0$, $d_5 \neq 0$, and $c_1d_2 - c_2d_1 \neq 0$. Suppose f has an infinite sequence of stable, single-round, periodic solutions accumulating on Γ_{HC} . Then*

$$d_2 = 0, \quad (7)$$

$$|\lambda\sigma| = 1, \quad (8)$$

$$|d_1| = 1, \quad (9)$$

with $d_1 = 1$ in the case $\lambda\sigma = 1$. Moreover, if $\lambda\sigma = 1$ and T_0 has the form (4) then

$$a_1 + b_1 = 0. \quad (10)$$

The equation $d_2 = 0$ corresponds to a homoclinic tangency, as shown in Fig. 1. With $|\lambda\sigma| = 1$ we have $|\det(Df)| = 1$ at the origin. The condition $d_1 = 1$ is a global condition, see [10] for a geometric interpretation, with which the tangency is termed globally resonant. Finally if $\lambda\sigma = 1$ and T_0 has the form (4), then $\det(Df) = 1 + 2(a_1 + b_1)xy + \mathcal{O}(x^2y^2)$. Thus the condition $a_1 + b_1 = 0$ implies that as (x, y) is varied from $(0, 0)$, the value of $\det(Df)$ varies quadratically instead of linearly (as is generically the case).

Now suppose $|\lambda\sigma| = 1$. Given $k_{\min} \in \mathbb{Z}$, let

$$K(k_{\min}) = \left\{ k \in \mathbb{Z} \mid k \geq k_{\min}, (\lambda\sigma)^k = d_1 \right\}. \quad (11)$$

If $\lambda\sigma = d_1 = 1$, then K is the set of all integers greater than or equal to $k_{\min}$. If $\lambda\sigma = -1$ and $d_1 = 1$ [resp. $d_1 = -1$], then K is the set of all even [resp. odd] integers greater than or equal to $k_{\min}$. Also let

$$\Delta = (1 - c_2 - d_1d_4)^2 - 4d_5(d_3 + c_1d_1). \quad (12)$$

Theorem 3.2. *Suppose f satisfies (3) with (4) and (5). Suppose $c_0 = 1$, $d_0 = 0$, and $d_5 \neq 0$. Suppose (7)–(10) are satisfied, $\Delta > 0$, and*

$$-1 < c_2 < 1 - \frac{\sqrt{\Delta}}{2}. \quad (13)$$

Then there exists $k_{\min} \in \mathbb{Z}$ such that f has an asymptotically stable period- $(k + m)$ solution for all $k \in K(k_{\min})$.

Theorems 3.1 and 3.2 imply that the phenomenon of infinitely many asymptotically stable, single-round periodic solutions is codimension-three in the case $\lambda\sigma = -1$. Specifically the three independent codimension-one conditions (7)–(9) need to hold, and the phenomenon indeed occurs if $\Delta > 0$ and (13) holds. In the case $\lambda\sigma = 1$ the phenomenon is codimension-four as we also require (10). This condition is absent when $\lambda\sigma = -1$ because in this case the cubic terms in (4) are removable.

4. Smooth parameter dependence. Now suppose f_μ is a C^∞ map on $\mathbb{R}^2$ with a C^∞ dependence on a parameter $\mu \in \mathbb{R}^n$. Let $\mathbf{0} \in \mathbb{R}^n$ denote the origin in parameter space. Suppose that for all μ in some region containing $\mathbf{0}$, the origin in phase space $(x, y) = (0, 0)$ is a fixed point of f_μ . Let $\lambda = \lambda(\mu)$ and $\sigma = \sigma(\mu)$ be its associated eigenvalues (these are C^∞ functions of μ) and suppose

$$\lambda(\mathbf{0}) = \alpha, \quad (14)$$

$$\sigma(\mathbf{0}) = \frac{\chi_{\text{eig}}}{\alpha}, \quad (15)$$

with $0 < \alpha < 1$ and $\chi_{\text{eig}} \in \{-1, 1\}$. With $\mu = \mathbf{0}$ we have $|\lambda\sigma| = 1$, so as described above T_0 can be assumed to have the form (4). We now show we can assume T_0 has this form when the value of μ is sufficiently small.

Lemma 4.1. *There exists a neighbourhood $\mathcal{N}_{\text{param}} \subset \mathbb{R}^n$ of $\mathbf{0}$ and a C^∞ coordinate change that puts f_μ in the form (4) for all $\mu \in \mathcal{N}_{\text{param}}$.*

Proof. Via a linear transformation f_μ can be transformed to

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} \lambda x + \sum_{i \geq 0, j \geq 0, i+j \geq 2} a_{ij} x^i y^j \\ \sigma y + \sum_{i \geq 0, j \geq 0, i+j \geq 2} b_{ij} x^i y^j \end{bmatrix},$$

for some $a_{ij}, b_{ij} \in \mathbb{R}$. It is a standard asymptotic matching exercise to show that via an additional C^∞ coordinate change we can achieve $a_{ij} = 0$ if $\lambda^{i-1}\sigma^j \neq 1$, and $b_{ij} = 0$ if $\lambda^i\sigma^{j-1} \neq 1$. The remainder of the proof is based on this fact.

Assume the value of μ is small enough that (1) is satisfied. Then $\lambda^p\sigma^q = 1$ is only possible with $p, q \geq 1$, so a C^∞ coordinate change can be performed to reduce the map to

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} \lambda x + \sum_{i \geq 2, j \geq 1} a_{ij} x^i y^j \\ \sigma y + \sum_{i \geq 1, j \geq 2} b_{ij} x^i y^j \end{bmatrix}.$$

Since $|\lambda\sigma| = 1$ when $\mu = \mathbf{0}$ we can assume μ is small enough that $\lambda^{p-1}\sigma \neq 1$ for all $p \geq 3$ and $\lambda\sigma^{q-1} \neq 1$ for all $q \geq 3$. Consequently the map can further be reduced to

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} \lambda x + a_{21}x^2y + \sum_{i \geq 3, j \geq 2} a_{ij}x^i y^j \\ \sigma y + b_{12}xy^2 + \sum_{i \geq 2, j \geq 3} b_{ij}x^i y^j \end{bmatrix},$$

which can be rewritten as (4). $\square$

The product of the eigenvalues is

$$\lambda(\mu)\sigma(\mu) = \chi_{\text{eig}} + \mathbf{n}_{\text{eig}}^T \mu + \mathcal{O}(\|\mu\|^2), \quad (16)$$

where $\mathbf{n}_{\text{eig}}$ is the gradient of $\lambda\sigma$ evaluated at $\mu = \mathbf{0}$. The following result is an elementary application of the implicit function theorem.

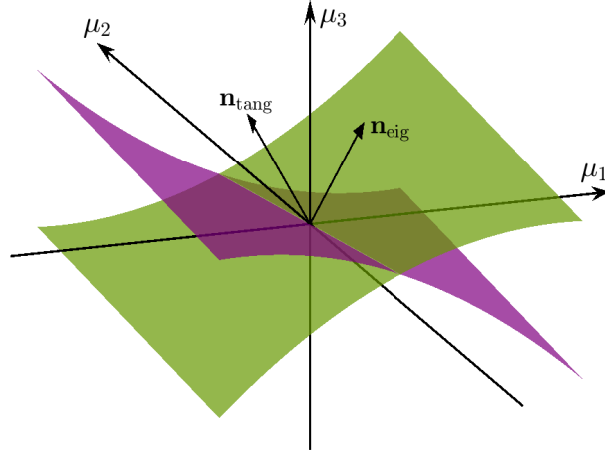


FIGURE 2. A sketch of codimension-one surfaces of homoclinic tangencies (green) and where $\lambda(\mu)\sigma(\mu) = 1$ (purple). The vectors $\mathbf{n}_{\text{tang}}$ and $\mathbf{n}_{\text{eig}}$, respectively, are normal to these surfaces at the origin $\mu = \mathbf{0}$.

Lemma 4.2. *Suppose $\mathbf{n}_{\text{eig}} \neq \mathbf{0}$. Then $|\lambda(\mu)\sigma(\mu)| = 1$ on a C^∞ codimension-one surface intersecting $\mu = \mathbf{0}$ and with normal vector $\mathbf{n}_{\text{eig}}$ at $\mu = \mathbf{0}$ (as illustrated in Fig. 2).*

5. The codimension-one surface of homoclinic tangencies. In this section we describe the codimension-one surface of homoclinic tangencies that intersects $\mu = \mathbf{0}$ where we will be assuming that a globally resonant homoclinic tangency occurs.

Suppose (5) is satisfied when $\mu = \mathbf{0}$. Write f_μ^m as (6) and suppose

$$c_0(\mathbf{0}) = 1, \quad (17)$$

$$d_0(\mathbf{0}) = 0, \quad (18)$$

so that f_0 has an orbit homoclinic to the origin through $(x, y) = (1, 0)$ and $(0, 1)$. Also suppose

$$d_2(\mathbf{0}) = 0, \quad (19)$$

$$d_5(\mathbf{0}) = d_{5,0} \neq 0, \quad (20)$$

for a quadratic tangency. Also write

$$d_0(\mu) = \mathbf{n}_{\text{tang}}^\top \mu + \mathcal{O}(\|\mu\|^2). \quad (21)$$

Lemma 5.1. *Suppose (5) is satisfied when $\mu = \mathbf{0}$, (17)–(20) are satisfied, and $\mathbf{n}_{\text{tang}} \neq \mathbf{0}$. Then f_μ has a quadratic homoclinic tangency to $(x, y) = (0, 0)$ on a C^∞ codimension-one surface intersecting $\mu = \mathbf{0}$ and with normal vector $\mathbf{n}_{\text{tang}}$ at $\mu = \mathbf{0}$. Moreover, this tangency occurs at $(x, y) = (0, Y(\mu))$ where Y is a C^∞ function with $Y(\mathbf{0}) = 1$.*

Proof. Let $T_{1,2}$ denote the second component of T_1 (6). The function

$$h(u; \mu) = \frac{\partial T_{1,2}}{\partial y}(0, 1 + u) = d_2(\mu) + 2d_5(\mu)u + \mathcal{O}(2),$$

is a C^∞ function of $u \in \mathbb{R}$ and μ . Since $h(0; \mathbf{0}) = 0$ by (19) and $\frac{\partial h}{\partial u}(0; \mathbf{0}) \neq 0$ by (20), the implicit function theorem implies there exists a C^∞ function $u_{\text{tang}}(\mu)$ such that $h(u_{\text{tang}}(\mu); \mu) = 0$ locally.

By construction, the unstable manifold of $(x, y) = (0, 0)$ is tangent to the x -axis at $T_1(0, 1 + u_{\text{tang}})$. Moreover this tangency is quadratic by (20). Thus a homoclinic tangency occurs if $T_{1,2}(0, 1 + u_{\text{tang}}) = d_0 + d_2 u_{\text{tang}} + d_5 u_{\text{tang}}^2 + \mathcal{O}(3) = 0$. This function is C^∞ and

$$T_{1,2}(0, 1 + u_{\text{tang}}) = \mathbf{n}_{\text{tang}}^\top \mu + \mathcal{O}(\|\mu\|^2).$$

Since $\mathbf{n}_{\text{tang}} \neq \mathbf{0}$ the result follows from another application of the implicit function theorem. $\square$

6. Sequences of saddle-node and period-doubling bifurcations. Suppose

$$d_1(\mathbf{0}) = \chi, \quad (22)$$

where $\chi \in \{-1, 1\}$. Write

$$\begin{aligned} a_1(\mathbf{0}) &= a_{1,0}, & c_1(\mathbf{0}) &= c_{1,0}, & d_3(\mathbf{0}) &= d_{3,0}, \\ b_1(\mathbf{0}) &= b_{1,0}, & c_2(\mathbf{0}) &= c_{2,0}, & d_4(\mathbf{0}) &= d_{4,0}, \end{aligned} \quad (23)$$

and, recalling (12), let

$$\Delta_0 = (1 - c_{2,0} - \chi d_{4,0})^2 - 4d_{5,0}(d_{3,0} + \chi c_{1,0}). \quad (24)$$

Theorem 6.1. *Suppose f_μ satisfies (14), (15), (17)–(20), $a_{1,0} + b_{1,0} = 0$, $\Delta_0 > 0$, and $-1 < c_{2,0} < 1 - \frac{\sqrt{\Delta_0}}{2}$. Let $\mathbf{v} \in \mathbb{R}^n$. If $\mathbf{n}_{\text{tang}}^\top \mathbf{v} \neq 0$ then there exists $k_{\min} \in \mathbb{Z}$ such that for all $k \in K(k_{\min})$ there exist $\varepsilon_k^- < 0$ and $\varepsilon_k^+ > 0$ with $\varepsilon_k^\pm = \mathcal{O}(\alpha^{2k})$ such that f has an asymptotically stable period- $(k+m)$ solution for all $\mu = \varepsilon \mathbf{v}$ with $\varepsilon_k^- < \varepsilon < \varepsilon_k^+$. Moreover, one sequence, $\{\varepsilon_k^-\}$ or $\{\varepsilon_k^+\}$, corresponds to saddle-node bifurcations of the periodic solutions, the other to period-doubling bifurcations. If instead $\mathbf{n}_{\text{tang}}^\top \mathbf{v} = 0$ and $\mathbf{n}_{\text{eig}}^\top \mathbf{v} \neq 0$ the same results hold except $\varepsilon_k^\pm = \mathcal{O}\left(\frac{\alpha^k}{k}\right)$.*

7. Proof of the main result. To prove Theorem 6.1 we use the following lemma which is proved in Appendix A by carefully estimating the error terms in (7.1). If $\chi_{\text{eig}} = -1$ then (25) is true if $a_1 = b_1 = 0$ (and this can be proved in the same fashion).

Lemma 7.1. *Suppose $\chi_{\text{eig}} = 1$ and $\mu = \mathcal{O}(\alpha^k)$. If $|x - 1|$ and $|\alpha^{-k}y - 1|$ are sufficiently small for all sufficiently large values of k , then*

$$T_0^k(x, y) = \left[\lambda^k x (1 + ka_1 xy + \mathcal{O}(k^2 \alpha^{2k})) \right] \left[\sigma^k y (1 + kb_1 xy + \mathcal{O}(k^2 \alpha^{2k})) \right]. \quad (25)$$

Proof of Theorem 6.1. Step 1 — Coordinate changes to distinguish the surface of homoclinic tangencies.

First we perform two coordinate changes on parameter space. There exists an $n \times n$ orthogonal matrix A such that after μ is replaced with $A\mu$, and μ is scaled, we have $\mathbf{n}_{\text{tang}}^\top = [1, 0, \dots, 0]$ — the first coordinate vector of $\mathbb{R}^n$. Then $d_0(\mu) = \mu_1 + \mathcal{O}(\|\mu\|^2)$. Second, for convenience, we apply a near-identity transformation to remove the higher order terms, resulting in

$$d_0(\mu) = \mu_1. \quad (26)$$

Importantly, these coordinate changes do not alter the signs of the dot products $\mathbf{n}_{\text{tang}}^T \mathbf{v}$ and $\mathbf{n}_{\text{eig}}^T \mathbf{v}$. Now write

$$c_0(\mu) = 1 + \sum_{i=1}^n p_i \mu_i + \mathcal{O}(\|\mu\|^2), \quad (27)$$

$$d_1(\mu) = \chi + \sum_{i=1}^n q_i \mu_i + \mathcal{O}(\|\mu\|^2), \quad (28)$$

$$d_2(\mu) = \sum_{i=1}^n r_i \mu_i + \mathcal{O}(\|\mu\|^2), \quad (29)$$

$$\lambda(\mu) = \alpha + \sum_{i=1}^n s_i \mu_i + \mathcal{O}(\|\mu\|^2), \quad (30)$$

$$\sigma(\mu) = \frac{\chi_{\text{eig}}}{\alpha} + \sum_{i=1}^n t_i \mu_i + \mathcal{O}(\|\mu\|^2), \quad (31)$$

where $p_i, \dots, t_i \in \mathbb{R}$ are constants.

Step 2 — Apply a k -dependent scaling to μ .

In view of the coordinate changes applied in the previous step, the surface of homoclinic tangencies of Lemma 5.1 is tangent to the $\mu_1 = 0$ coordinate hyperplane. In order to show that bifurcation values scale like $|\alpha|^{2k}$ if we adjust the value of μ in a direction transverse to this surface, and, generically, scale like $\frac{|\alpha|^k}{k}$ otherwise, we scale the components of μ as follows:

$$\mu_i = \begin{cases} \alpha^{2k} \tilde{\mu}_i, & i = 1, \\ \alpha^k \tilde{\mu}_i, & i \neq 1. \end{cases} \quad (32)$$

Below we will see that the resulting asymptotic expansions are consistent and this will justify (32). Notice that $\tilde{\mu}_1$ -terms are higher order than $\tilde{\mu}_i$ -terms, for $i \neq 1$. For example (30) now becomes $\lambda = \alpha + \sum_{i=2}^n s_i \tilde{\mu}_i \alpha^k + \mathcal{O}(|\alpha|^{2k})$. Further, let k be such that $\lambda(\mathbf{0})^k \sigma(\mathbf{0})^k = d_1(\mathbf{0})$, that is $\chi_{\text{eig}}^k = \chi$. Then from (30)–(32) we obtain

$$\lambda^k = \alpha^k \left(1 + \frac{k}{\alpha} \sum_{i=2}^n s_i \tilde{\mu}_i \alpha^k + \mathcal{O}(k^2 |\alpha|^{2k}) \right), \quad (33)$$

$$\sigma^k = \frac{\chi}{\alpha^k} \left(1 + k \alpha \chi_{\text{eig}} \sum_{i=2}^n t_i \tilde{\mu}_i \alpha^k + \mathcal{O}(k^2 |\alpha|^{2k}) \right). \quad (34)$$

Step 3 — Calculate one point of each periodic solution.

One point of a single-round periodic solution is a fixed point of $T_0^k \circ T_1$. We look for fixed points $(x^{(k)}, y^{(k)})$ of $T_0^k \circ T_1$ of the form

$$\begin{aligned} x^{(k)} &= \alpha^k (1 + \phi_k \alpha^k + \mathcal{O}(k^2 |\alpha|^{2k})), \\ y^{(k)} &= 1 + \psi_k \alpha^k + \mathcal{O}(k^2 |\alpha|^{2k}). \end{aligned} \quad (35)$$

By substituting (35) into (6) and the above various asymptotic expressions for the coefficients in T_1 , we obtain

$$T_1(x^{(k)}, y^{(k)})$$

$$= \begin{bmatrix} 1 + (c_{1,0} + c_{2,0}\psi_k + \sum_{i=2}^n p_i \tilde{\mu}_i) \alpha^k + \mathcal{O}(k^2 |\alpha|^{2k}) \\ \chi \alpha^k + \left(\tilde{\mu}_1 + d_{3,0} + d_{4,0}\psi_k + d_{5,0}\psi_k^2 + \chi \phi_k + \sum_{i=2}^n (q_i + r_i \psi_k) \tilde{\mu}_i \right) \alpha^{2k} \\ + \mathcal{O}(k^2 |\alpha|^{3k}) \end{bmatrix}.$$

Then by (25),

$$(T_0^k \circ T_1)(x^{(k)}, y^{(k)}) = \begin{bmatrix} \alpha^k + \left(a_{1,0}\chi + \frac{1}{\alpha} \sum_{i=2}^n s_i \tilde{\mu}_i \right) k \alpha^{2k} + \left(c_{1,0} + c_{2,0}\psi_k + \sum_{i=2}^n p_i \tilde{\mu}_i \right) \alpha^{2k} \\ + \mathcal{O}(k^2 |\alpha|^{3k}) \\ 1 + \left(b_{1,0}\chi + \alpha \chi_{\text{eig}} \sum_{i=2}^n t_i \tilde{\mu}_i \right) k \alpha^k + \left(\chi \tilde{\mu}_1 + \chi d_{3,0} + \chi d_{4,0}\psi_k \right. \\ \left. + \chi d_{5,0}\psi_k^2 + \phi_k + \chi \sum_{i=2}^n (q_i + r_i \psi_k) \tilde{\mu}_i \right) \alpha^k + \mathcal{O}(k^2 |\alpha|^{2k}) \end{bmatrix}. \quad (36)$$

By matching (35) and (36) and eliminating ϕ_k we obtain the following expression that determines the possible values of ψ_k :

$$\chi d_{5,0} \psi_k^2 - P \psi_k + Q = 0, \quad (37)$$

where

$$P = 1 - c_{2,0} - \chi d_{4,0} - \chi \sum_{i=2}^n r_i \tilde{\mu}_i, \quad (38)$$

$$Q = \chi \tilde{\mu}_1 + c_{1,0} + \chi d_{3,0} + \sum_{i=2}^n (p_i + \chi q_i) \tilde{\mu}_i + \sum_{i=2}^n \left(\frac{s_i}{\alpha} + \alpha \chi_{\text{eig}} t_i \right) \tilde{\mu}_i k, \quad (39)$$

and we have also used $a_{1,0} + b_{1,0} = 0$. Of the two solutions to (37), the one that yields an asymptotically stable solution when $\mu = \mathbf{0}$ is

$$\psi_k = \frac{1}{2\chi d_{5,0}} \left(P - \sqrt{P^2 - 4\chi d_{5,0} Q} \right). \quad (40)$$

Note that this solution exists and is real-valued for sufficiently small values of μ because when $\mu = \mathbf{0}$ the discriminant is $P^2 - 4\chi d_{5,0} Q = \Delta_0 > 0$.

Step 4 — Stability of the periodic solution.

By using (6), (25), (33), (34), and (35),

$$D(T_0^k \circ T_1)(x^{(k)}, y^{(k)}) = \begin{bmatrix} \mathcal{O}(|\alpha|^k) & c_{2,0} \alpha^k + \mathcal{O}(k |\alpha|^{2k}) \\ \frac{1}{\alpha^k} (1 + \mathcal{O}(k |\alpha|^k)) & \chi d_{4,0} + \chi \sum_{i=2}^n r_i \tilde{\mu}_i \\ & + 2\chi d_{5,0} \psi_k + \mathcal{O}(k |\alpha|^k) \end{bmatrix}. \quad (41)$$

Let τ_k and δ_k denote the trace and determinant of this matrix, respectively. By (38), (40), and (41) we obtain

$$\tau_k = 1 - c_{2,0} - \sqrt{P^2 - 4\chi d_{5,0}Q} + \mathcal{O}(k|\alpha|^k), \quad (42)$$

$$\delta_k = -c_{2,0} + \mathcal{O}(k|\alpha|^k). \quad (43)$$

By substituting $\mu = \mathbf{0}$ into (42) we obtain $\tau_k = 1 - c_{2,0} - \sqrt{\Delta_0}$. It immediately follows from the assumption $-1 < c_{2,0} < 1 - \frac{\sqrt{\Delta_0}}{2}$ that the periodic solution is asymptotically stable for sufficiently large values of k .

Step 5 — The generic case $\mathbf{n}_{\text{tang}}^\top \mathbf{v} \neq 0$.

Now suppose $\mathbf{n}_{\text{tang}}^\top \mathbf{v} \neq 0$, that is, $v_1 \neq 0$ (in view of the earlier coordinate change). Write $\mu = \varepsilon \mathbf{v}$ and $\varepsilon = \tilde{\varepsilon} \alpha^{2k}$. By (32), $\tilde{\mu}_1 = \tilde{\varepsilon} v_1$ and $\tilde{\mu}_i = \tilde{\varepsilon} v_i \alpha^k$ for $i \neq 1$. Then by (38) and (39), $P = 1 - c_{2,0} - \chi d_{4,0} + \mathcal{O}(|\alpha|^k)$ and $Q = c_{1,0} + \chi d_{3,0} + \chi \tilde{\varepsilon} v_1 + \mathcal{O}(|\alpha|^k)$, so

$$P^2 - 4\chi d_{5,0}Q = (1 - c_{2,0} - \chi d_{4,0})^2 - 4\chi d_{5,0}(c_{1,0} + \chi d_{3,0} + \chi \tilde{\varepsilon} v_1) + \mathcal{O}(|\alpha|^k). \quad (44)$$

Since $v_1 \neq 0$ and $d_{5,0} \neq 0$ we can solve $\delta_k - \tau_k + 1 = 0$ for $\tilde{\varepsilon}$ (formally this is achieved via the implicit function theorem) and the solution is

$$\tilde{\varepsilon}_{\text{SN}} = \frac{\Delta_0}{4d_{5,0}v_1} + \mathcal{O}(k|\alpha|^k). \quad (45)$$

Also we can use (44) to solve $\delta_k + \tau_k + 1 = 0$ for $\tilde{\varepsilon}$:

$$\tilde{\varepsilon}_{\text{PD}} = \frac{\Delta_0 - 4(1 - c_{2,0})^2}{4d_{5,0}v_1} + \mathcal{O}(k|\alpha|^k). \quad (46)$$

Since $\tilde{\varepsilon}_{\text{SN}}$ and $\tilde{\varepsilon}_{\text{PD}}$ evidently have opposite signs, this completes the proof in the first case.

Step 6 — The degenerate case $\mathbf{n}_{\text{tang}}^\top \mathbf{v} = 0$.

Now suppose $v_1 = 0$ and $\mathbf{n}_{\text{eig}}^\top \mathbf{v} \neq 0$. Again write $\mu = \varepsilon \mathbf{v}$ but now write $\varepsilon = \frac{\tilde{\varepsilon} \alpha^k}{k}$. By (32), $\tilde{\mu}_1 = 0$ and $\tilde{\mu}_i = \frac{\tilde{\varepsilon} v_i}{k}$ for $i \neq 1$.

We first evaluate $\mathbf{n}_{\text{eig}}^\top \mathbf{v}$. By multiplying (30) and (31) and comparing the result to (16) we see that the i^{th} element of $\mathbf{n}_{\text{eig}}$ is $\frac{\chi_{\text{eig}} s_i}{\alpha} + \alpha t_i$. Then since $v_1 = 0$ we have $\mathbf{n}_{\text{eig}}^\top \mathbf{v} = \sum_{i=2}^n \left(\frac{\chi_{\text{eig}} s_i}{\alpha} + \alpha t_i \right) v_i$. Further, by $\mu = \varepsilon \mathbf{v}$ and $\varepsilon = \frac{\tilde{\varepsilon} \alpha^k}{k}$,

$$\chi_{\text{eig}} \tilde{\varepsilon} \mathbf{n}_{\text{eig}}^\top \mathbf{v} = \sum_{i=2}^n \left(\frac{s_i}{\alpha} + \alpha \chi_{\text{eig}} t_i \right) \tilde{\mu}_i k,$$

which is a term appearing in (39). So by (38) and (39), $P = 1 - c_{2,0} - \chi d_{4,0} + \mathcal{O}(\frac{1}{k})$ and $Q = c_{1,0} + \chi d_{3,0} + \frac{\tilde{\varepsilon} \chi_{\text{eig}} \mathbf{n}_{\text{eig}}^\top \mathbf{v}}{k} + \mathcal{O}(\frac{1}{k})$. By solving $\delta_k - \tau_k + 1 = 0$,

$$\tilde{\varepsilon}_{\text{SN}} = \frac{\Delta_0}{4d_{5,0}\chi\chi_{\text{eig}}\mathbf{n}_{\text{eig}}^\top \mathbf{v}} + \mathcal{O}\left(\frac{1}{k}\right), \quad (47)$$

and by solving $\delta_k + \tau_k + 1 = 0$,

$$\tilde{\varepsilon}_{\text{PD}} = \frac{\Delta_0 - 4(1 - c_{2,0})^2}{4d_{5,0}\chi\chi_{\text{eig}}\mathbf{n}_{\text{eig}}^\top \mathbf{v}} + \mathcal{O}\left(\frac{1}{k}\right). \quad (48)$$

As in the previous case, $\tilde{\varepsilon}_{\text{SN}}$ and $\tilde{\varepsilon}_{\text{PD}}$ have opposite signs, and this completes the proof in the second case. Notice how the assumptions we have made ensure the denominators of (47) and (48) are nonzero. $\square$

8. A comparison to numerically computed bifurcation values. Here we extend the example given in [10] and illustrate the results with the following C^1 family of maps

$$f(x, y) = \begin{cases} U_0(x, y), & y \leq \frac{2\alpha+1}{3}, \\ (1-r(y))U_0(x, y) + r(y)U_1(x, y), & \frac{2\alpha+1}{3} \leq y \leq \frac{\alpha+2}{3}, \\ U_1(x, y), & y \geq \frac{\alpha+2}{3}, \end{cases} \quad (49)$$

where

$$U_0(x, y) = \begin{bmatrix} (\alpha + \mu_2)x(1 + (a_{1,0} + \mu_4)xy) \\ \frac{1}{\alpha}y(1 - a_{1,0}xy) \end{bmatrix}, \quad (50)$$

$$U_1(x, y) = \begin{bmatrix} 1 + c_{2,0}(y-1) \\ \mu_1 + (1 + \mu_3)x + d_{5,0}(y-1)^2 \end{bmatrix}, \quad (51)$$

$$r(y) = s\left(\frac{3y - (2\alpha + 1)}{1 - \alpha}\right), \quad (52)$$

where

$$s(z) = 3z^2 - 2z^3. \quad (53)$$

The function r has been designed so that f is C^1 on the switching lines $y = \frac{2\alpha+1}{3}$ and $y = \frac{\alpha+2}{3}$, although this is not particularly important because we only wish to describe single-round periodic solutions near the critical homoclinic orbit, and this orbit is bounded away from the switching lines. Below we fix

$$\begin{aligned} \alpha &= 0.8, \\ a_{1,0} &= 0.2, \\ c_{2,0} &= -0.5, \\ d_{5,0} &= 1, \end{aligned} \quad (54)$$

and vary $\mu = (\mu_1, \mu_2, \mu_3, \mu_4) \in \mathbb{R}^4$.

With $\mu = \mathbf{0}$, (49) satisfies the conditions of Theorem 3.2. In particular $\Delta_0 = 2.25$ so $\Delta_0 > 0$ and $-1 < c_{2,0} < 1 - \frac{\Delta_0}{2}$. Therefore (49) has an asymptotically stable single-round periodic solution for all sufficiently large values of k . In fact these exist for all $k \geq 1$, see Fig. 3, plus there exists an asymptotically stable fixed point at $(x, y) = (1, 1)$ that can be interpreted as corresponding to $k = 0$.

In the remainder of this section we study how the infinite coexistence is destroyed by varying each the components of μ from zero in turn. We identify saddle-node and period-doubling bifurcations numerically and compare these to our above asymptotic results.

First, by varying the value of μ_1 from zero we destroy the homoclinic tangency. Indeed in (21) we have $\mathbf{n}_{\text{tang}}^T = [1, 0, 0, 0]$. Thus if we fix $\mu_2 = \mu_3 = \mu_4 = 0$ and vary the value of μ_1 , by Theorem 6.1 there must exist sequences of saddle-node and period-doubling bifurcations occurring at values of μ_1 that are asymptotically proportional to α^{2k} . Fig. 4a shows the bifurcation values (obtained numerically) for six different values of k . We have designed (49) so that it satisfies (26). Consequently the formulas (45) and (46) for the bifurcation values can be applied directly. In panel (b) we observe that the numerically computed bifurcation values indeed converge to their leading-order approximations.

We now fix $\mu_1 = \mu_3 = \mu_4 = 0$ and vary the value of μ_2 . This parameter variation alters the product of the eigenvalues associated with the origin. Specifically $\mathbf{n}_{\text{eig}}^T =$

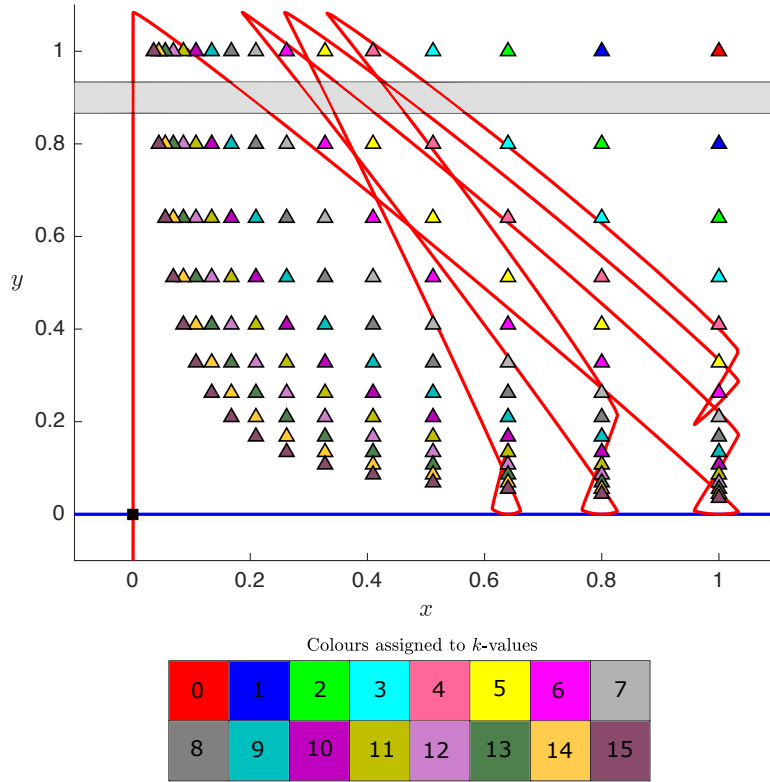


FIGURE 3. A phase portrait of (49) with (54) and $\mu = \mathbf{0}$. The shaded horizontal strip is where the middle component of (49) applies. We show parts of the stable and unstable manifolds of $(x, y) = (0, 0)$. Note the unstable manifold has very high curvature at $(x, y) \approx (0, 1.1)$ because (49) is highly nonlinear in the horizontal strip. For the given parameter values (49) has an asymptotically stable, single-round periodic solutions of period $k + 1$ for all $k \geq 1$. These are shown for $k = 1, 2, \dots, 15$; different colours correspond to different values of k . The map also has an asymptotically stable fixed point at $(x, y) = (1, 1)$.

$[0, \frac{1}{\alpha}, 0, 0]$ in (16) so by Theorem 6.1 the bifurcation values are asymptotically proportional to $\frac{\alpha^k}{k}$. In Fig. 5 we see the numerically computed bifurcation values converging to their leading order approximations (47) and (48).

Next we fix $\mu_1 = \mu_2 = \mu_4 = 0$ and vary the value of μ_3 which breaks the global resonance condition. Here $\mathbf{n}_{\text{tang}}^T \mathbf{v} = 0$ and $\mathbf{n}_{\text{eig}}^T \mathbf{v} = 0$ so Theorem (6.1) does not apply. But by performing calculations analogous to those given above in the proof of Theorem 6.1 directly to the map (49), we obtain the following expressions for the saddle-node and period-doubling bifurcation values

$$\epsilon_{\text{SN}} = \frac{(1 - c_{2,0})^2}{4d_{5,0}} \alpha^k + \mathcal{O}(|\alpha|^{2k}), \quad (55)$$

$$\epsilon_{\text{PD}} = -\frac{3(1 - c_{2,0})^2}{4d_{5,0}}\alpha^k + \mathcal{O}(|\alpha|^{2k}). \quad (56)$$

Fig. 6 shows that the numerically computed bifurcations do indeed appear to be converging to the leading-order components of (55) and (56). Notice the bifurcation values are asymptotically proportional to α^k (a slightly slower rate than that in Fig. 5).

Finally we fix $\mu_1 = \mu_2 = \mu_3 = 0$ and vary the value of μ_4 which breaks the condition on the resonance terms in T_0 . As in the previous case Theorem 6.1 does

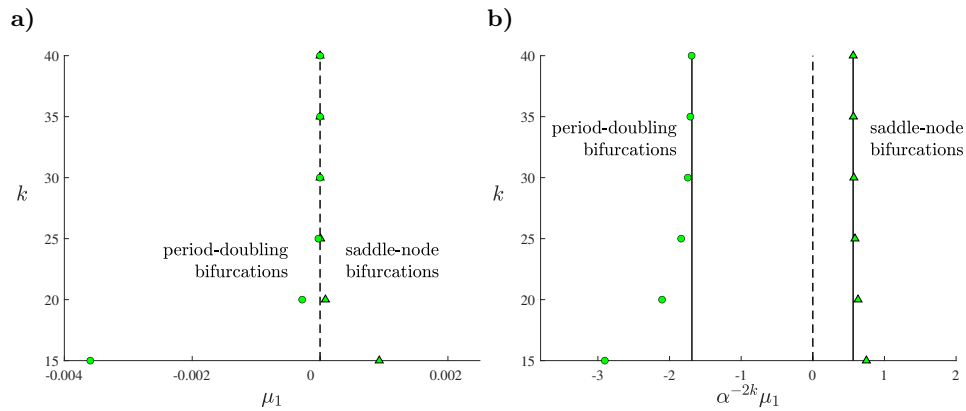


FIGURE 4. Panel (a) is a numerically computed bifurcation diagram of (49) with (54) and $\mu_2 = \mu_3 = \mu_4 = 0$. The triangles [circles] are saddle-node [period-doubling] bifurcations of single-round periodic solutions of period $k + 1$. Panel (b) shows the same points but with the horizontal axis scaled in such a way that the asymptotic approximations to these bifurcations, given by the leading-order terms in (45) and (46), appear as vertical lines.

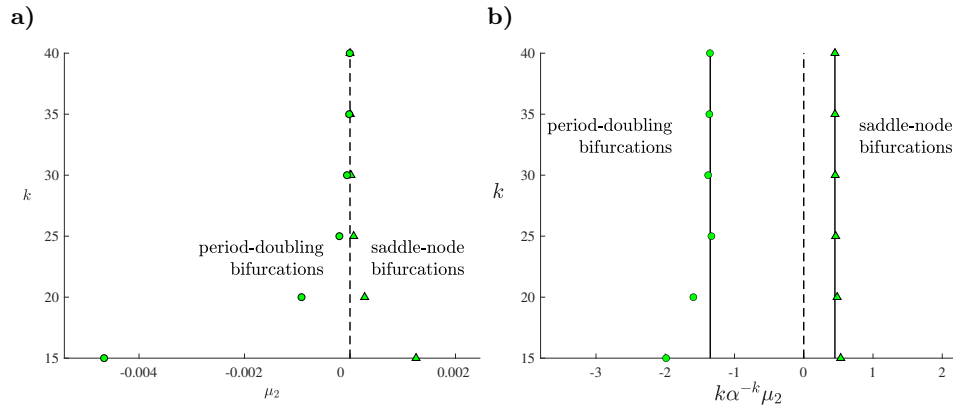


FIGURE 5. Panel (a) is a numerically computed bifurcation diagram of (49) with (54) and $\mu_1 = \mu_3 = \mu_4 = 0$. Panel (b) shows convergence to the leading-order terms of (47) and (48).

not apply. By again calculating the bifurcations as above we obtain

$$\epsilon_{\text{SN}} = \frac{(1 - c_{2,0})^2}{4d_{5,0}k} + \mathcal{O}\left(\frac{1}{k^2}\right), \quad (57)$$

$$\epsilon_{\text{PD}} = -\frac{3(1 - c_{2,0})^2}{4d_{5,0}k} + \mathcal{O}\left(\frac{1}{k^2}\right), \quad (58)$$

and these agree with the numerically computed bifurcation values as shown in Fig. 7. The bifurcation values are asymptotically proportional to $\frac{1}{k}$ which is substantially slower than in the previous three cases.

Together these scaling results show that in four-dimensional parameter space the stability region between surfaces of saddle-node and period-doubling bifurcations becomes narrower in different directions at different rates as $k \rightarrow \infty$. This is partially illustrated by Fig. 8 where we have fixed $\mu_3 = \mu_4 = 0$. As k increases the regions become more narrow in the μ_1 direction than the μ_2 direction and approach

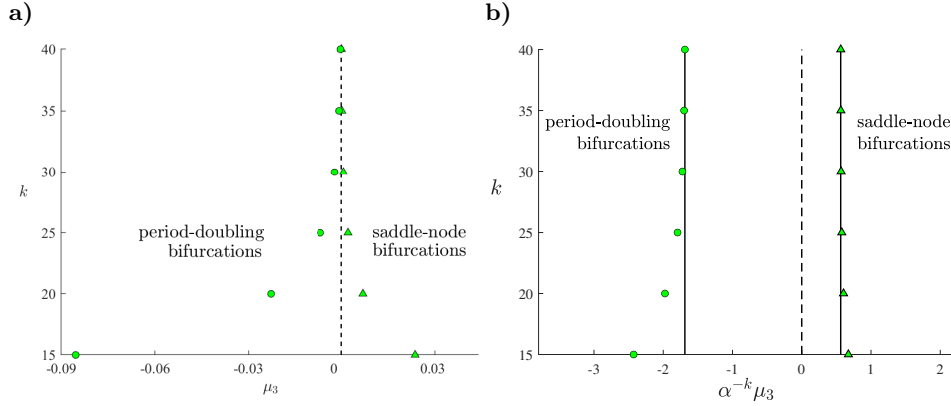


FIGURE 6. Panel (a) is a numerically computed bifurcation diagram of (49) with (54) and $\mu_1 = \mu_2 = \mu_4 = 0$. Panel (b) shows convergence to the leading-order terms of (55) and (56).

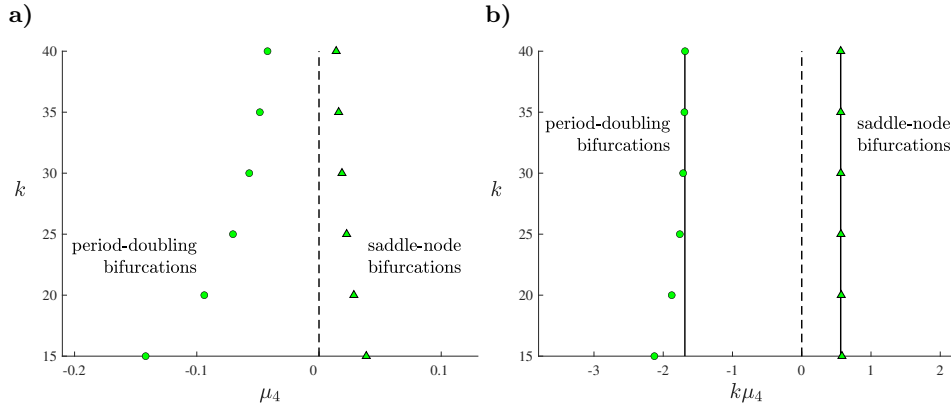


FIGURE 7. Panel (a) is a numerically computed bifurcation diagram of (49) with (54) and $\mu_1 = \mu_2 = \mu_3 = 0$. Panel (b) shows convergence to the leading-order terms of (57) and (58).

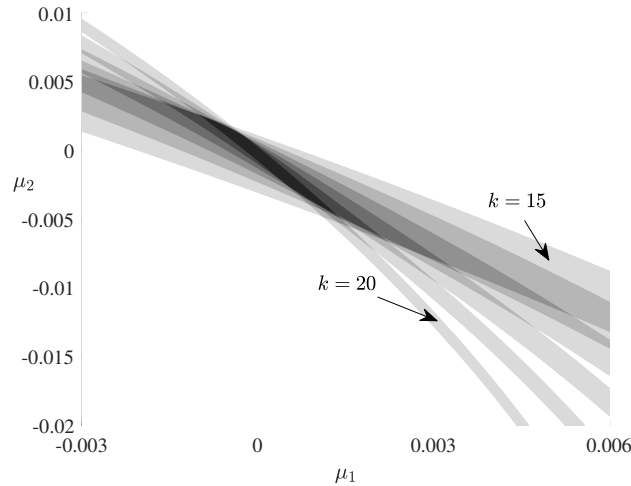


FIGURE 8. A two-dimensional slice of the four-dimensional parameter space of (49) defined by fixing $\mu_3 = \mu_4 = 0$. The remaining parameter values are given by (54) except we have set $a_{1,0} = 0$ to simplify the numerical computations. For each $15 \leq k \leq 20$ we show the region bounded by curves of saddle-node and period-doubling bifurcations where (49) has an asymptotically stable period- $(k+1)$ solution. Intersections of these regions are indicated by successively darker shades of grey.

the line $\mu_1 = 0$ because μ_1 is the coordinate of fastest convergence. Stable periodic solutions for different values of k coexist in areas where these regions intersect. For the example shown in Fig. 8, all six regions intersect in the black area which is bounded by nine of the 12 bifurcation boundaries. For different values of the parameters we have observed that a different number of bifurcation curves can bound the black area. This exemplifies the complexity possible for the shapes of regions where some fixed number of stable solutions coexist.

9. Discussion. In this paper we have considered globally resonant homoclinic tangencies in smooth two-dimensional maps and determined scaling laws for the size of parameter intervals in which single-round periodic solutions are asymptotically stable. We have illustrated the results with an abstract four-parameter family. It remains to identify globally resonant homoclinic tangencies in prototypical maps and maps derived from physical applications.

About a parameter point satisfying the conditions of Theorem 3.2, for any positive $j \in \mathbb{Z}$ there exists an open region of parameter space in which the family of maps has j coexisting asymptotically stable periodic solutions. It follows from Theorem 6.1 that the largest ball (sphere) that the region contains has a diameter asymptotically proportional to $|\alpha|^{2j}$. But, as we have shown, different directions of perturbation yield different scaling laws. Consequently we expect such regions to have an elongated shape for large values of j , and this is hinted at by the two-dimensional slice of parameter space shown in Fig. 8. This reveals that such regions may have a particularly complicated shape, bounded by many of the saddle-node

and period-doubling bifurcations identified above. We believe the primary $|\alpha|^{2k}$ scaling law holds true for higher-dimensional maps. Certainly similar aspects of homoclinic tangencies have been shown to be independent of dimension [7]. In the piecewise-linear setting, globally resonant homoclinic tangencies were analysed in higher dimensions in [16].

Appendix A. Proof of Lemma 7.1. Write

$$T_0(x, y) = \begin{bmatrix} \lambda x \left(1 + a_1 xy + x^2 y^2 \tilde{F}(x, y) \right) \\ \sigma y \left(1 + b_1 xy + x^2 y^2 \tilde{G}(x, y) \right) \end{bmatrix}. \quad (59)$$

Let $R > 0$ be such that

$$|a_1|, |b_1|, |\tilde{F}(x, y)|, |\tilde{G}(x, y)| \leq R, \quad (60)$$

for all $(x, y) \in \mathcal{N}$ and all sufficiently small values of μ . For simplicity we assume $\alpha > 0$; if instead $\alpha < 0$ the proof can be completed in the same fashion.

We have $\lambda = \alpha + \mathcal{O}(|\alpha|^k)$ and $\sigma = \frac{1}{\alpha} + \mathcal{O}(|\alpha|^k)$. Thus there exists $M \geq 2R$ such that $\lambda \leq \alpha(1 + M\alpha^k)$ and $\sigma \leq \frac{1}{\alpha}(1 + M\alpha^k)$ for sufficiently large values of k . It follows (by induction on j) that

$$\sigma^j \leq \alpha^{-j} (1 + 2Mj\alpha^k), \quad (61)$$

and

$$(\lambda\sigma)^j \leq 1 + 4Mj\alpha^k, \quad (62)$$

for all $j = 1, 2, \dots, k$ again assuming k is sufficiently large.

Let $\varepsilon > 0$. Assume

$$|x - 1| \leq \varepsilon, \quad |\alpha^{-k}y - 1| \leq \varepsilon, \quad (63)$$

for sufficiently large values of k . We can assume $\varepsilon < 1 - \frac{1}{\sqrt{2}}$ so then

$$\frac{\alpha^k}{2} \leq |xy| \leq 2\alpha^k. \quad (64)$$

Write

$$T_0^j(x, y) = \begin{bmatrix} \lambda^j x \left(1 + ja_1 xy + x^2 y^2 \tilde{F}_j(x, y) \right) \\ \sigma^j y \left(1 + jb_1 xy + x^2 y^2 \tilde{G}_j(x, y) \right) \end{bmatrix}. \quad (65)$$

Below we will use induction on j to show that

$$\left| \tilde{F}_j(x, y) \right|, \left| \tilde{G}_j(x, y) \right| \leq 12Mj^2, \quad (66)$$

for all $j = 1, 2, \dots, k$, assuming k is sufficiently large. This will complete the proof because with $j = k$, (66) implies (25).

Clearly (66) is true for $j = 1$: $\left| \tilde{F}_1(x, y) \right| = \left| \tilde{F}(x, y) \right| \leq R \leq \frac{M}{2} < 12M$ and similarly for $\tilde{G}_1$. Suppose (66) is true for some $j < k$. It remains for us to verify (66) for $j + 1$. First observe that by using $|a_1| \leq R$, (64), and the induction hypothesis,

$$\left| 1 + ja_1 xy + x^2 y^2 \tilde{F}_j(x, y) \right| \leq 1 + 2Rj\alpha^k + 48Mj^2\alpha^{2k}.$$

For sufficiently large k this implies

$$\left| 1 + ja_1 xy + x^2 y^2 \tilde{F}_j(x, y) \right| \leq 1 + 2Mj\alpha^k, \quad (67)$$

where we have also used $M \geq 2R$. Similarly

$$\left| 1 + jb_1xy + x^2y^2\tilde{G}_j(x, y) \right| \leq 1 + 2Mj\alpha^k. \quad (68)$$

Write $T_0^j(x, y) = (x_j, y_j)$. By using (61), (63), and (68) we obtain

$$|y_j| \leq \alpha^{k-j}(1 + \varepsilon)(1 + 2Mj\alpha^k)^2.$$

Thus $|y_j| \leq \alpha^{k-j}(1 + 2\varepsilon)$, say, for sufficiently large values of k . Also $|x_jy_j|$ is clearly small, so we can conclude that $(x_j, y_j) \in \mathcal{N}$ (in particular we have shown that (x_{k-1}, y_{k-1}) can be made as close to $(0, \frac{1}{\alpha})$ as we like).

By matching the first components of $T_0^{j+1}(x, y) = (T_0 \circ T_0^j)(x, y)$ we obtain

$$\begin{aligned} \lambda^{j+1}x \left(1 + (j+1)a_1xy + x^2y^2\tilde{F}_{j+1}(x, y) \right) &= \lambda^{j+1}x \left(1 + ja_1xy + x^2y^2\tilde{F}_j(x, y) \right) \\ &\quad + \lambda^{j+1}a_1x^2y(1 + P) + \lambda^{j+1}x^3y^2Q, \end{aligned} \quad (69)$$

where

$$\begin{aligned} 1 + P &= (\lambda\sigma)^j \left(1 + ja_1xy + x^2y^2\tilde{F}_j(x, y) \right)^2 \left(1 + jb_1xy + x^2y^2\tilde{G}_j(x, y) \right), \\ Q &= (\lambda\sigma)^{2j} \left(1 + ja_1xy + x^2y^2\tilde{F}_j(x, y) \right)^3 \left(1 + jb_1xy + x^2y^2\tilde{G}_j(x, y) \right)^2 \tilde{F}(x_j, y_j). \end{aligned} \quad (71)$$

By (62), (67), and (68), we obtain

$$\begin{aligned} 1 + P &\leq (1 + 4Mj\alpha^k)(1 + 2Mj\alpha^k)^3 \\ &\leq 1 + 11Mj\alpha^k, \\ Q &\leq (1 + 4Mj\alpha^k)^2(1 + 2Mj\alpha^k)^5R \\ &\leq 2R, \end{aligned}$$

assuming k is sufficiently large and where we have also used $|\tilde{F}(x_j, y_j)| \leq R$ (valid because $(x_j, y_j) \in \mathcal{N}$). From (69),

$$\tilde{F}_{j+1}(x, y) = \tilde{F}_j(x, y) + \frac{P}{xy} + Q.$$

Then by using the induction hypothesis, the lower bound on $|xy|$ (64), and the above bounds on P and Q , we arrive at

$$\begin{aligned} \left| \tilde{F}_{j+1}(x, y) \right| &\leq 12Mj^2 + 22Mj + 2R \\ &\leq 12Mj^2 + 24Mj \\ &< 12M(j+1)^2. \end{aligned}$$

In a similar fashion by matching the second components of $T_0^{j+1}(x, y) = (T_0 \circ T_0^j)(x, y)$ we obtain $\left| \tilde{G}_{j+1}(x, y) \right| < 12M(j+1)^2$. This verifies (66) for $j+1$ and so completes the proof. $\square$

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E-mail address: S.Muni@massey.ac.nz

E-mail address: R.McLachlan@massey.ac.nz

E-mail address: D.J.W.Simpson@massey.ac.nz