



Robust Chaos in Orientation-Reversing and Non-Invertible Two-Dimensional Piecewise-Linear Maps

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Abstract

The two-dimensional border-collision normal form is a four-parameter family of piecewise-linear maps generalizing the Lozi family and relevant to diverse applications. It was recently shown to exhibit a chaotic attractor throughout an open region of parameter space, but only for parameter combinations where the map is invertible and orientation-preserving. In this paper we formally identify analogous regions for the orientation-reversing and non-invertible cases, the latter being particularly relevant to control systems. This is achieved by constructing a trapping region in phase space and an invariant expanding cone in tangent space, and provides a more complete picture of robust chaos by showing its presence is disassociated from the global topological properties of the map. We also show some boundaries of the regions are optimal in that they correspond to bifurcations where the chaotic attractor is destroyed.

Keywords Discrete dynamics · Piecewise-smooth · Chaotic attractor · Invariant cone · Border-collision bifurcation

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1 Introduction

Robust chaos refers to the persistence of a chaotic attractor throughout open regions of the parameter space of a family of dynamical systems (Glendinning 2017). Such robustness is essential for chaos-based cryptography (Kocarev and Lian 2011) and can be helpful to devices that have advantageous functional characteristics when operated in a chaotic regime, examples include power converters (Deane and Hamill 1996), optical resonators (Liu et al. 2013), and energy harvesters (Kumar et al. 2016).

Robust chaos occurs in diverse settings (Gonchenko et al. 2018; Zeraoulia and Sprott 2012) and its occurrence in piecewise-linear maps was popularized by the 1998 paper of Banerjee et al. (1998). Later Glendinning and Simpson (2021) formally identified *explicit* parameter values for which robust chaos occurs, in contrast to earlier results that give *implicit* conditions in more abstract settings (Pesin 1992; Rychlik 2004; Young 1985). However, Glendinning and Simpson (2021) only dealt with two-dimensional, orientation-preserving maps. Piecewise-linear maps that are not orientation-preserving are arguably more physically relevant than those that are orientation-preserving, as discussed below. The purpose of this paper is to formally identify robust chaos in the non-orientation-preserving cases, which is also an important precursor to demonstrating robust chaos in higher dimensional maps.

Specifically we study the two-dimensional *border-collision normal form*

$$f_{\xi}(x, y) = \begin{cases} (\tau_L x + y + 1, -\delta_L x), & x \leq 0, \\ (\tau_R x + y + 1, -\delta_R x), & x \geq 0. \end{cases} \quad (1)$$

For any fixed parameter point $\xi = (\tau_L, \delta_L, \tau_R, \delta_R) \in \mathbb{R}^4$, we are interested in the behavior of iterations $(x, y) \mapsto f_{\xi}(x, y)$ on $\mathbb{R}^2$. The family (1) is continuous but non-differentiable on the line $x = 0$, termed the *switching manifold*, and is a normal form in the sense that any two-dimensional, piecewise-linear, continuous map with a single switching manifold and satisfying a genericity condition can be transformed to (1) through a change of coordinates (Simpson 2022, Appendix A). The form (1) was originally formulated by Nusse and Yorke (1992) with an additional parameter μ that when varied through zero brings about a border-collision bifurcation (Banerjee and Grebogi 1999). The simplicity of (1) belies the incredible complexity of its dynamics; it can exhibit two-dimensional attractors (Glendinning 2016; Glendinning and Wong 2011), large numbers of coexisting attractors (Pumariño et al. 2018; Simpson 2014, 2022), and Arnold tongues with a unique sausage-string geometry (Simpson 2017). In the special case $\tau_L = -\tau_R$ and $\delta_L = \delta_R$, (1) reduces to the Lozi family (Lozi 1978).

At any point with $x \neq 0$ the derivative of (1) has determinant

$$\det((Df_{\xi})(x, y)) = \begin{cases} \delta_L, & x < 0, \\ \delta_R, & x > 0. \end{cases}$$

Consequently we have the following cases.

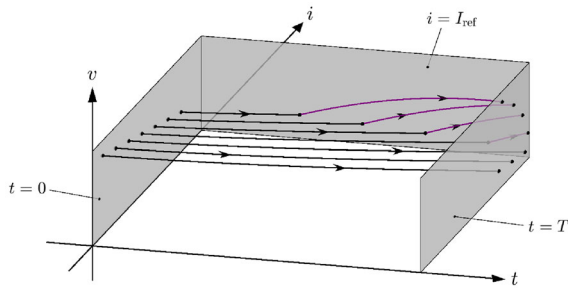


Fig. 1 Typical trajectories of the boost converter model of Deane (1992). The two variables $i(t)$ (current) and $v(t)$ (voltage) obey certain linear ODEs, except if the current reaches the threshold I_{ref} then a switch opens and the variables obey different ODEs until time T where the commutation cycle ends, the switch is closed, and the process repeats. The dynamics are characterized by the stroboscopic map taking $(i(0), v(0))$ to $(i(T), v(T))$. As seen from the paths of the trajectories (colored black where the switch is closed and purple where the switch is open), this map is piecewise-smooth and non-invertible. Refer to (Banerjee et al. 1998; Deane 1992; Fatoyinbo and Simpson 2023) for more details (color figure online)

- If $\delta_L > 0$ and $\delta_R > 0$ then f_ξ is orientation-preserving. This is the most natural case to consider from an applied perspective because Poincaré maps derived from time-reversible flows on $\mathbb{R}^n$ are necessarily orientation-preserving (Wiggins 1988).
- If $\delta_L < 0$ and $\delta_R < 0$ then f_ξ is orientation-reversing. This case was considered in the seminal work of Misiurewicz (1980) for the Lozi map. Such maps can be embedded into higher dimensional orientation-preserving maps and in this way help us understand high-dimensional chaos in physical systems (Karatetskaia et al. 2021).
- If δ_L and δ_R have opposite signs then f_ξ is non-invertible with the left and right half-planes mapping onto either the top or the bottom half-plane. Such maps, except piecewise-smooth instead of piecewise-linear, apply to a wide range of power converters (Banerjee and Verghese 2001; di Bernardo and Tse 2002; Zhusubaliyev and Mosekilde 2003). For instance, the boost converter described by Banerjee et al. (1998), and earlier by Deane (1992), regulates output voltage via a switch that is activated whenever the current reaches a threshold value. This causes the flow (in phase space) to fold back over itself resulting in a stroboscopic map that is piecewise-smooth (due to the switch) and non-invertible (due to the folding) (Fatoyinbo and Simpson 2023), Fig. 1.
- If one of δ_L and δ_R is zero then f_ξ is non-invertible with one half-plane mapping onto a line (Kowalczyk 2005). Such maps arise as leading-order approximations to grazing-sliding bifurcations in relay control systems (di Bernardo et al. 2002) and mechanical systems with stick-slip friction (di Bernardo et al. 2003; Szalai and Osinga 2008). This is because trajectories of the differential equations become constrained to a codimension-one discontinuity surface causing the range of one piece of the Poincaré map to have one less dimension than the domain of the map (di Bernardo et al. 2002). This also occurs for the Hicksian trade cycle model of Puu and Sushko (2006) and the influenza outbreak model of Roberts et al. (2019).

Let

$$\Phi = \left\{ \xi \in \mathbb{R}^4 \mid \tau_L > |\delta_L + 1|, \tau_R < -|\delta_R + 1| \right\} \quad (2)$$

denote the set of all ξ for which f_ξ has two saddle fixed points (Lemma 2.1). Banerjee et al. (1998) considered $\xi \in \Phi$ in the orientation-preserving case and showed that on one side of a homoclinic bifurcation the stable manifold of one fixed point has transverse intersections with the unstable manifolds of both fixed points and argued that this implies the existence of a chaotic attractor. Their conclusion was verified rigorously by Glendinning and Simpson (2021) using the methodology of Misiurewicz (1980). First, a forward invariant region $\Omega \subset \mathbb{R}^2$ was identified by using one of the unstable manifolds, and this was perturbed into a *trapping region* that necessarily contains a topological attractor. Second, an *invariant expanding cone* $\Psi \subset T\mathbb{R}^2$ (see Sect. 4) was constructed by using eigenvectors of the two pieces of Df_ξ . The existence of this object implies that nearby forward orbits diverge exponentially and that the attractor is chaotic in the sense of having a positive Lyapunov exponent. In general, this approach works brilliantly for piecewise-linear maps because the invariance and expansion properties can be verified by simple, explicit computations. For more complicated constructions the verification is best done on a computer (Glendinning and Simpson 2022). The construction is robust to nonlinear perturbations to the pieces of the map and has been used to show that chaos persists for intervals of parameter values beyond border-collision bifurcations (Simpson and Glendinning 2021). Recently this approach has also been applied to piecewise-smooth maps with a square-root singularity that arise for mechanical systems with hard impacts (Miao et al. 2019). Further techniques can reveal more properties of the attractor, such as sensitive dependence on initial conditions (Ghosh and Simpson 2022), continuity with respect to ξ (Alves et al. 2017; Kucharski 2023), and the presence of an SRB measure (Glendinning 2017).

In this paper we verify robust chaos in the orientation-reversing and non-invertible cases. We obtain a subset $\Phi_{\text{trap}} \subset \Phi$ in which f_ξ has a trapping region and another subset $\Phi_{\text{cone}} \subset \Phi$ in which f_ξ has an invariant expanding cone. It follows that f_ξ has a chaotic attractor for all $\xi \in \Phi_{\text{trap}} \cap \Phi_{\text{cone}}$. This is an open subset of parameter space, hence the chaos is robust with respect to the family f_ξ . We expect it is also robust to nonlinear perturbations to the pieces of the map, as in Simpson and Glendinning (2021). Boundaries of $\Phi_{\text{trap}} \cap \Phi_{\text{cone}}$ are where some aspect of the construction fails. As shown below, three of these boundaries correspond to bifurcations where the chaotic attractor is destroyed. Beyond the other boundaries the chaotic attractor appears to persist and we believe robust chaos could be verified on a larger subset of parameter space by using a more complicated construction, e.g., Simpson (2022), but our aim here is not to optimize the subset, only to obtain a reasonably large subset for both negative and positive values of δ_L and δ_R by using a construction that is both natural and simple.

The remainder of this paper is organized as follows. We start in Sect. 2 by calculating the stable and unstable manifolds of the fixed points. Constraints on the geometry of the manifolds as they are extended outwards from the fixed points give rise to our definition of Φ_{trap} . Here we also define Φ_{cone} and state our main result (Theorem 2.2) for the existence of a chaotic attractor. Being four-dimensional the sets Φ_{trap} and Φ_{cone} are difficult to visualize; we show a range of two-dimensional cross-sections to give some impression of their size and shape.

In Sect. 3, we use the stable manifold of one of the fixed points to form a region $\Omega \subset \mathbb{R}^2$ and show it is forward invariant under f_ξ for any $\xi \in \Phi_{\text{trap}}$. We then show

how Ω can be perturbed into a trapping region for any $\xi \in \Phi_{\text{trap}}$. In Sec. 4 we identify a cone that is both invariant and expanding for any $\xi \in \Phi_{\text{cone}}$ and prove Theorem 2.2. In Sect. 5, we further study the extent to which Φ_{trap} and Φ_{cone} cover parameter space. We show a typical cross-section of $\Phi_{\text{trap}} \cap \Phi_{\text{cone}}$ and overlay numerical simulations whereby the presence of robust chaos is estimated from forward orbits. This helps clarify which boundaries of Φ_{trap} and Φ_{cone} correspond to bifurcations and how much of the true robust chaos region is detected by the straightforward constructions. Final remarks are provided in Sect. 6.

2 Sufficient Conditions for a Chaotic Attractor

For any $\xi \in \Phi$ the map f_ξ has fixed points

$$X = \left(\frac{-1}{\tau_R - \delta_R - 1}, \frac{\delta_R}{\tau_R - \delta_R - 1} \right), \quad (3)$$

$$Y = \left(\frac{-1}{\tau_L - \delta_L - 1}, \frac{\delta_L}{\tau_L - \delta_L - 1} \right), \quad (4)$$

where X is in the right half-plane and Y is in the left half-plane. The stability multipliers associated with X and Y are the eigenvalues of the Jacobian matrices

$$A_L = \begin{bmatrix} \tau_L & 1 \\ -\delta_L & 0 \end{bmatrix}, \quad A_R = \begin{bmatrix} \tau_R & 1 \\ -\delta_R & 0 \end{bmatrix}. \quad (5)$$

Since $\xi \in \Phi$, A_L has eigenvalues $\lambda_L^s \in (-1, 1)$ and $\lambda_L^u > 1$, while A_R has eigenvalues $\lambda_R^s \in (-1, 1)$ and $\lambda_R^u < -1$. This implies X and Y are saddles. In fact, Φ is the set of all parameter combinations for which f_ξ has two saddle fixed points:

Lemma 2.1 *The map f_ξ has two saddle fixed points if and only if $\xi \in \Phi$.*

Proof Let f^L and f^R denote the left and right pieces of f_ξ , respectively. If $\delta_L = \tau_L - 1$ then f^L has no fixed points, while if $\delta_L \neq \tau_L - 1$ then f^L has the unique fixed point Y , given by (4). Similarly if $\delta_R = \tau_R - 1$ then f^R has no fixed points, while if $\delta_R \neq \tau_R - 1$ then f^R has the unique fixed point X , given by (3).

If $\xi \in \Phi$ then X and Y are saddle fixed points of f_ξ , as noted above. Conversely, suppose f_ξ has two saddle fixed points. By the above remarks, these must be X and Y . Since they are fixed points of f_ξ , Y is in the left half-plane, so $\tau_L > \delta_L + 1$ by (4), and X is in the right half-plane, so $\tau_R < \delta_R + 1$ by (3). Since they are saddles, also $\tau_L > -(\delta_L + 1)$ and $\tau_R < -(\delta_R + 1)$, hence $\xi \in \Phi$. $\square$

2.1 The Stable and Unstable Manifolds of the Fixed Points

Since X and Y are saddles they have one-dimensional stable and unstable manifolds. Their stable manifolds $W^s(X)$ and $W^s(Y)$ have kinks at points where they meet the switching manifold $x = 0$ and on preimages of these points, while their unstable

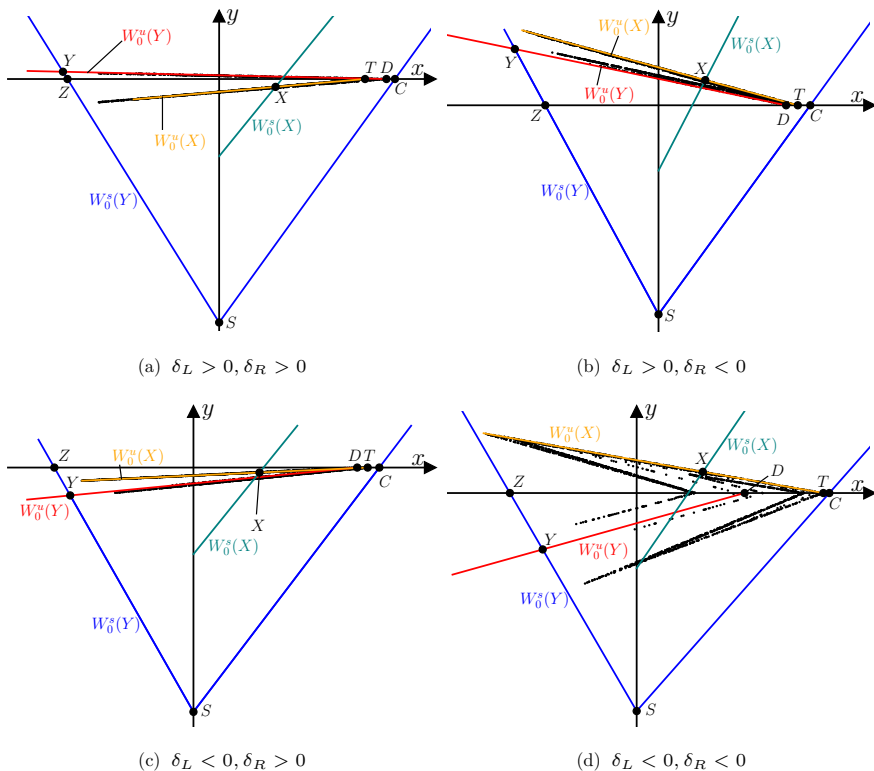


Fig. 2 Phase portraits of f_{ξ} (1) for four different parameter combinations $\xi = (\tau_L, \delta_L, \tau_R, \delta_R) \in \Phi_{\text{trap}} \cap \Phi_{\text{cone}}$: **a** $\xi = (2.1, 0.06, -1.7, 0.18)$; **b** $\xi = (2.1, 0.4, -1.7, -0.55)$; **c** $\xi = (2.2, -0.3, -1.7, 0.1)$; **d** $\xi = (1.8, -0.75, -1.6, -0.4)$. Each plot shows the initial line segments of the stable and unstable manifolds emanating from the fixed points X and Y , as well as an additional segment of the stable manifold of Y . We also show some key points of intersection (S , C , D , T , and Z) of these segments with the coordinate axes, see (6), (9), (11), (12), and (23), respectively. To illustrate the chaotic attractor the black dots show iterates of a typical forward orbit after transient dynamics has decayed

manifolds $W^u(X)$ and $W^u(Y)$ have kinks at points where they meet $y = 0$ (the image of the switching manifold) and on images of these points. As subsets of $\mathbb{R}^2$ these manifolds have extremely complicated geometries (necessarily so when the map is chaotic), but for our purposes it suffices to consider the first few linear parts of the manifolds as they emanate from the fixed points. Specifically we use some points where the manifolds intersect $x = 0$ and $y = 0$ for our constructions in Sect. 3.

For each manifold the first linear part coincides with the line through the fixed point with direction given by the corresponding eigenvector of A_L or A_R . We will use a subscript 0 to denote the part of the manifold that coincides with this line. These are indicated in Fig. 2 for four different combinations of the parameter values.

For all $\xi \in \Phi$, $W_0^s(Y)$ has an endpoint on $x = 0$, call it S . The slope of $W_0^s(Y)$ is that of the eigenvector of A_L corresponding to the stable eigenvalue λ_L^s . Due to the companion matrix form (5) this slope is given by $-\lambda_L^u$. Then from the above formula for Y we obtain

$$S = \left(0, \frac{-\lambda_L^u}{\lambda_L^u - 1} \right). \quad (6)$$

Notice $S_2 < -1$ (here and throughout the paper for any $P \in \mathbb{R}^2$ we write P_1 and P_2 , respectively, for its x and y components). The manifold $W^s(Y)$ continues into the right half-plane but now with slope $\frac{\phi_1(\xi)}{\lambda_L^u}$, where

$$\phi_1(\xi) = \delta_R - \tau_R \lambda_L^u. \quad (7)$$

Our trapping region construction requires that this linear segment, call it $W_1^s(Y)$, intersects $y = 0$. Certainly this is only possible if $\phi_1(\xi) > 0$. This inequality turns out to be sufficient to ensure $W_1^s(Y)$ intersects $y = 0$ except in the case $\delta_L, \delta_R < 0$ (Fig. 2d). In this case $W_1^s(Y)$ is the line segment connecting S and $f_\xi^{-2}(S)$. So in this case for $W_1^s(Y)$ to intersect $y = 0$ we need $f_\xi^{-2}(S)$ to lie above $y = 0$. A straightforward calculation reveals that the y -component of $f_\xi^{-2}(S)$ is $\frac{\phi_2(\xi)}{\lambda_L^s(\lambda_L^u - 1)\delta_R}$, where

$$\phi_2(\xi) = \delta_R(\lambda_L^s + 1) - \lambda_L^u(\tau_R + (\delta_R + \tau_R)\lambda_L^s). \quad (8)$$

Thus we require $\phi_2(\xi) > 0$ (because with $\delta_L, \delta_R < 0$ we have $\lambda_L^s(\lambda_L^u - 1)\delta_R > 0$). In any case, if $W_1^s(Y)$ intersects $y = 0$, it does so at the point

$$C = \left(1 + \frac{\phi_3(\xi)}{(\lambda_L^u - 1)\phi_1(\xi)}, 0 \right), \quad (9)$$

where

$$\phi_3(\xi) = \delta_R - (\delta_R + \tau_R - (\tau_R + 1)\lambda_L^u)\lambda_L^u. \quad (10)$$

Our construction also requires $C_1 > 1$, that is $\phi_3(\xi) > 0$.

Now we consider the two unstable manifolds. For all $\xi \in \Phi$, $W_0^u(Y)$ has an endpoint on $y = 0$ at

$$D = \left(\frac{1}{1 - \lambda_L^s}, 0 \right), \quad (11)$$

and $W_0^u(X)$ has an endpoint on $y = 0$ at

$$T = \left(\frac{1}{1 - \lambda_R^s}, 0 \right). \quad (12)$$

These points are indicated in Fig. 2.

2.2 Homoclinic and Heteroclinic Bifurcations

In the orientation-preserving case ($\delta_L, \delta_R > 0$), Banerjee et al. (1998) noticed that as parameters are varied a chaotic attractor can be destroyed when the points C and D collide. This occurs when the attractor contains orbits that converge to $W^u(Y)$ so includes points arbitrarily close to D as $C \rightarrow D$, hence beyond the bifurcation such orbits eventually land to the right of $W_1^s(Y)$ and subsequently diverge. This type of bifurcation is a *homoclinic corner* (Simpson 2016) where the kinks (corners) of the unstable manifold of Y lie on the stable manifold of Y (and vice-versa). This is analogous to a first homoclinic tangency (Palis and Takens 1993) for smooth maps which are well understood as a mechanism for the destruction of an attractor (Grebogi et al. 1983). From the above formulas for C and D we obtain

$$C_1 - D_1 = \frac{\phi_4(\xi)}{(\lambda_L^u - 1)(1 - \lambda_L^s)\phi_1(\xi)}, \quad (13)$$

where

$$\phi_4(\xi) = \delta_R - (\tau_R + \delta_L + \delta_R - (1 + \tau_R)\lambda_L^u)\lambda_L^u. \quad (14)$$

So the condition $\phi_4(\xi) > 0$ (equivalent to Eq. (5) of Banerjee et al. (1998)) ensures that C lies to the right of D as in Fig. 2a.

In the orientation-reversing case ($\delta_L, \delta_R < 0$) a chaotic attractor can extend close to T and consequently be destroyed in a similar fashion when the points C and T collide. Here kinks of the unstable manifold of X lie on the stable manifold of Y (and vice-versa). We have

$$C_1 - T_1 = \frac{\phi_5(\xi)}{(\lambda_L^u - 1)(1 - \lambda_R^s)\phi_1(\xi)}, \quad (15)$$

where

$$\phi_5(\xi) = \delta_R - (\delta_R + \tau_R - (1 + \lambda_R^u)\lambda_L^u)\lambda_L^u. \quad (16)$$

The condition $\phi_5(\xi) > 0$ ensures that C lies to the right of T as in Fig. 2-d. In the special case of the Lozi map, $\phi_5(\xi) > 0$ simplifies (significantly) to equation (3) of Misiurewicz (1980).

In view of the above discussion, we define

$$\Phi_{\text{trap}} = \{\xi \in \Phi \mid \phi_i(\xi) > 0, i = 1, \dots, 5\}. \quad (17)$$

This set is difficult to visualize because parameter space is four-dimensional. Figure 3 shows four different cross-sections of Φ_{trap} obtained by fixing τ_L and τ_R . Broadly speaking the size of the cross-section decreases as the values of τ_L and $|\tau_R|$ increase. Notice the topology of the cross-sections is different for different values of τ_L and τ_R . For instance in Fig. 3a, the boundary of the cross-section is formed by $\phi_1(\xi) = 0$, $\phi_2(\xi) = 0$, and the boundary of Φ , whereas in Fig. 3d the boundary is formed by

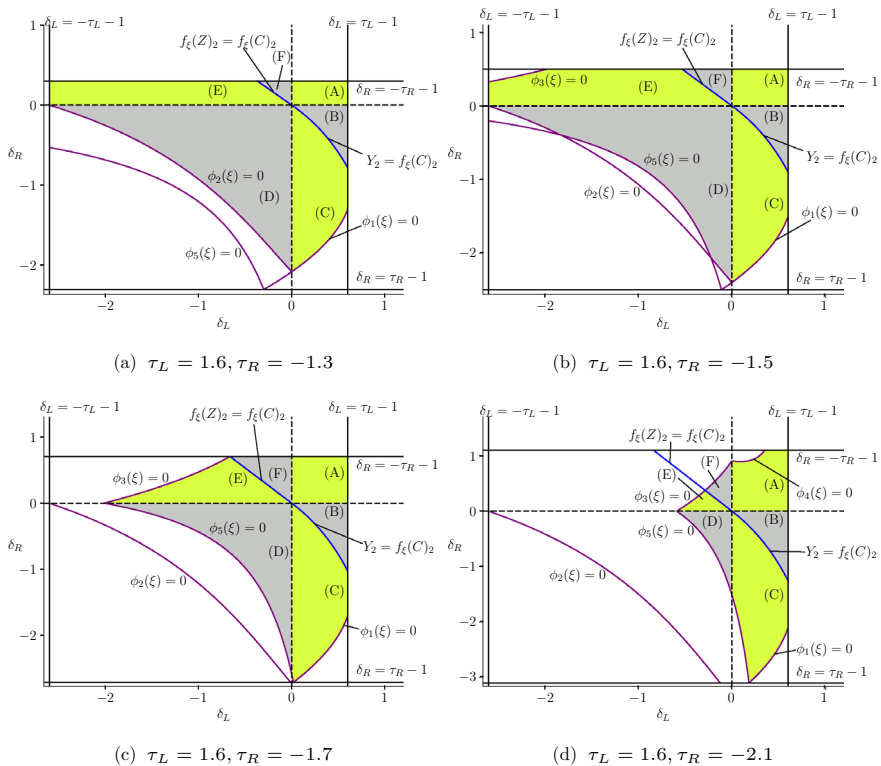


Fig. 3 Four different two-dimensional cross-sections of Φ_{trap} , defined by (17). In each plot, the cross-section is divided into six parts A–F where Ω has certain vertices, see Sect. 3

$\phi_1(\xi) = 0, \phi_3(\xi) = 0, \phi_4(\xi) = 0, \phi_5(\xi) = 0$, and the boundary of Φ . The figure also includes curves on which $f_\xi(C) = Y$ and $f_\xi(C) = Z$, where $Z = f_\xi(S)$ is the intersection of $W_0^s(Y)$ with $y = 0$. Together with the δ_L and δ_R axes, these curves divide the cross-sections into six parts corresponding to six cases for the vertices of the region Ω that we construct in Sect. 3.

2.3 Sufficient Conditions for Robust Chaos

Our construction of an invariant expanding cone Ψ_K requires similar constraints on the parameter values to those established above for the trapping region. To this end we define

$$\theta_1(\xi) = (\delta_L + \delta_R - \tau_L \tau_R)^2 - 4\delta_L \delta_R, \quad (18)$$

$$\theta_2(\xi) = \tau_L^2 + \delta_L^2 - 1 + 2\tau_L \min\left(0, -\frac{\delta_R}{\tau_R}, q_L, \tilde{a}\right), \quad (19)$$

$$\theta_3(\xi) = \tau_R^2 + \delta_R^2 - 1 + 2\tau_R \max\left(0, -\frac{\delta_L}{\tau_L}, q_R, \tilde{b}\right), \quad (20)$$

where

$$q_L = -\frac{\tau_L}{2} \left(1 - \sqrt{1 - \frac{4\delta_L}{\tau_L^2}} \right), \quad q_R = -\frac{\tau_R}{2} \left(1 - \sqrt{1 - \frac{4\delta_R}{\tau_R^2}} \right),$$

and

$$\tilde{a} = \frac{\delta_L - \delta_R - \tau_L \tau_R - \sqrt{\theta_1(\xi)}}{2\tau_R}, \quad \tilde{b} = \frac{\delta_R - \delta_L - \tau_L \tau_R - \sqrt{\theta_1(\xi)}}{2\tau_L},$$

assuming $\theta_1(\xi) > 0$. We then define

$$\Phi_{\text{cone}} = \{\xi \in \Phi \mid \theta_i(\xi) > 0, i = 1, \dots, 3\}. \quad (21)$$

The condition $\theta_1(\xi) > 0$ ensures $\tilde{a}$ and $\tilde{b}$ are well-defined, while the conditions $\theta_2(\xi) > 0$ and $\theta_3(\xi) > 0$ from the requirement that the cone Ψ_K is expanding. Specifically we need $\|A_L v\| \geq c\|v\|$ and $\|A_R v\| \geq c\|v\|$ for some $c > 1$ and all $v \in \Psi_K$, and by linearity it suffices to consider vectors of the form $v = \begin{bmatrix} 1 \\ m \end{bmatrix}$. From the formulas (5) for A_L and A_R , we have

$$\left\| A_J \begin{bmatrix} 1 \\ m \end{bmatrix} \right\|^2 - \left\| \begin{bmatrix} 1 \\ m \end{bmatrix} \right\|^2 = \tau_J^2 + \delta_J^2 - 1 + 2\tau_J m,$$

for each $J \in \{L, R\}$, which needs to be positive and the specific values of m in (19) and (20) correspond to the boundaries of Ψ_K . ensure that our cone Ψ_K .

Figure 4 shows cross-sections of Φ_{cone} . Broadly speaking the size of the cross-section *increases* as the values of τ_L and $|\tau_R|$ increase. Again the topology of the cross-sections is different for different values of τ_L and τ_R . Similar to Fig. 3, we have divided the cross-sections into six parts corresponding to six cases for the boundary of Ψ_K (see Sect. 4). These correspond to different cases for the four quantities in each of (19) and (20) that attain the minimum (respectively, maximum) value.

Finally, we can state our main result.

Theorem 2.2 *For any $\xi \in \Phi_{\text{trap}} \cap \Phi_{\text{cone}}$ the normal form f_ξ (1) has a topological attractor with a positive Lyapunov exponent.*

This is proved at the end of Sect. 4. We have found that there are many possibilities for the topology of cross-sections of $\Phi_{\text{trap}} \cap \Phi_{\text{cone}}$ defining by fixing τ_L and τ_R , and we do not attempt to categorize these in this paper. We provide one example in Sect. 5, where we also identify critical values of τ_L and τ_R at which the cross-sections of Φ_{trap} and Φ_{cone} vanish entirely.

3 A Forward Invariant Region and a Trapping Region

The following definition applies to any continuous map f on the plane. Note we write $\text{int}(\cdot)$ for the *interior* of a set.

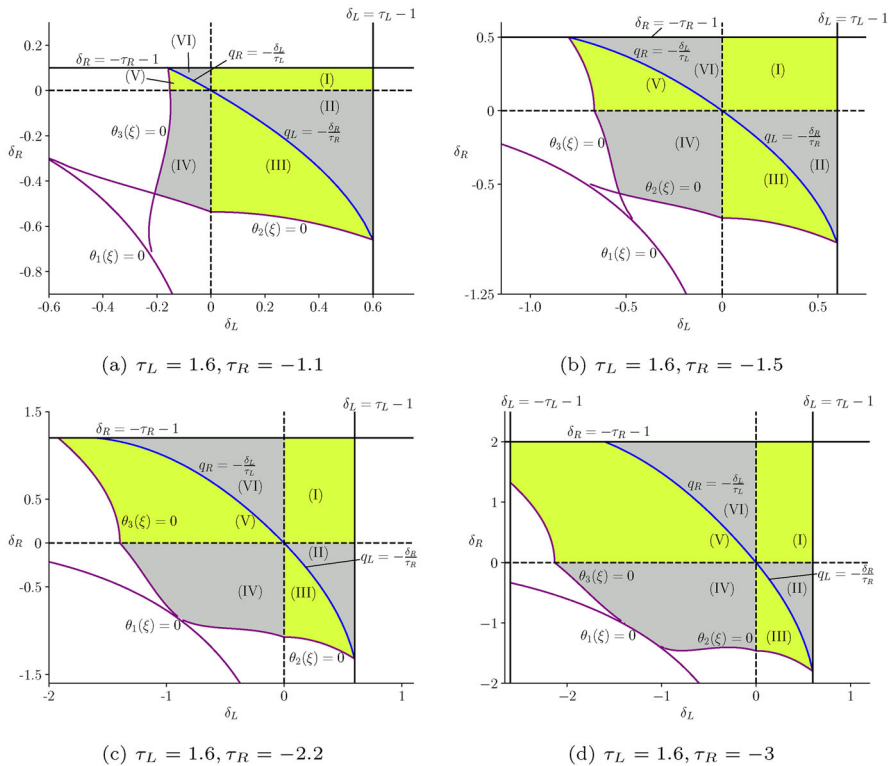


Fig. 4 Two-dimensional cross-sections of Φ_{cone} , defined by (21). In each plot the cross-section has been divided into six parts (I)–(VI) where the cone Ψ_K has different boundaries, see Fig. 7

Definition 3.1 A set $\Omega \subset \mathbb{R}^2$ is *forward invariant* if $f(\Omega) \subset \Omega$. A compact set $\Omega \subset \mathbb{R}^2$ is a *trapping region* if $f(\Omega) \subset \text{int}(\Omega)$.

In this section, we construct a triangle Ω and show that for any $\xi \in \Phi_{\text{trap}}$ this region is forward invariant under f (Proposition 3.4). We then show there exists a perturbation of Ω that is a trapping region for f (Proposition 3.5). So that the non-invertible cases can be accommodated, our Ω differs from the triangle constructed by Glendinning and Simpson (2021) for the orientation-preserving case, and by Misiurewicz (1980) for the orientation-reversing case. For clarity we now suppress the ξ -dependency and write f instead of f_ξ .

The linear segment of the stable manifold of Y that contains Y , denoted $W_0^s(Y)$, is shown in Fig. 2 for four different combinations of the parameter values. In any case, this segment lies on the line $y = -\lambda_L^u x + S_2$, where S_2 is the y -component of S , given by (6). The point S is the right-most point of $W_0^s(Y)$, and is easy to see that in the other direction $W_0^s(Y)$ extends to infinity if $\delta_L \geq 0$ and to the preimage of S under the left piece of f otherwise:

Lemma 3.2 Suppose $\delta_L \in \mathbb{R}$ and $\tau_L > |\delta_L + 1|$. Then

$$W_0^s(Y) = \begin{cases} \{(x, y) \mid -\frac{S_2}{\delta_L} \leq x \leq 0, y = -\lambda_L^u x + S_2\}, & \delta_L < 0, \\ \{(x, y) \mid -\infty < x \leq 0, y = -\lambda_L^u x + S_2\}, & \delta_L \geq 0, \end{cases} \quad (22)$$

and $f(W_0^s(Y)) \subset W_0^s(Y)$.

In particular $Z = f(S) \in W_0^s(Y)$. This point is the intersection of $W_0^s(Y)$ with $y = 0$ and given by

$$Z = \left(\frac{-1}{\lambda_L^u - 1}, 0 \right). \quad (23)$$

Now recall if $\phi_1(\xi) > 0$ and $\phi_2(\xi) > 0$ then $C \in W_1^s(Y)$ is given by (9).

Lemma 3.3 Let $\xi \in \Phi$ with $\phi_1(\xi) > 0$ and $\phi_2(\xi) > 0$. Then

$$Y, Z, f(Z), f(C), f^2(C) \in W_0^s(Y) \setminus \{S\}. \quad (24)$$

Proof Certainly $Y, Z \in W_0^s(Y) \setminus \{S\}$ by construction. Also $f(Z) \in W_0^s(Y) \setminus \{S\}$ because $W_0^s(Y)$ is forward invariant (Lemma 3.2) and $Z = f(S)$ cannot be a preimage of S .

Since $C_1 > 0$ we have $f(C) = (\tau_R C_1 + 1, -\delta_R C_1)$, and it is a simple exercise to use the formula (11) to show that $f(C)$ lies on the line $y = -\lambda_L^u x + S_2$. Also $f(C)$ lies to the left of S because $f(C)_1 = \frac{(\tau_R + \delta_R)(\lambda_L^u - 1) + \tau_R}{(\lambda_L^u - 1)\phi_1(\xi)}$ is negative by inspection, and in the case $\delta_L < 0$ the point $f(C)$ lies to the right of the left-most point of $W_0^s(Y)$ because $f(C)_1 + \frac{S_2}{\delta_L} = \frac{\phi_2(\xi)}{-\lambda_L^s(\lambda_L^u - 1)\phi_1(\xi)}$ is positive. Thus $f(C)$ belongs to $W_0^s(Y)$ and is not an endpoint of $W_0^s(Y)$, thus $f(C) \in W_0^s(Y) \setminus \{S\}$ and $f^2(C) \in W_0^s(Y) \setminus \{S\}$ using again Lemma 3.2. $\square$

Given $\xi \in \Phi$ with $\phi_1(\xi) > 0$ and $\phi_2(\xi) > 0$, let Q and R be the upper-most and lower-most points of $\{Y, Z, f(Z), f(C), f^2(C)\}$, respectively. Then let Ω be the compact filled triangle with vertices C, Q , and R (except Ω is a line segment in the special case $\delta_L = \delta_R = 0$). In other words, Ω is the convex hull of $Y, Z, f(Z), f(C), f^2(C)$, and C .

There are six cases for the points that form the vertices of Ω . These are shown in Fig. 5 and correspond to the six parts of Φ_{trap} indicated in Fig. 3. Figure 5 also shows the set $f(\Omega)$. Notice in each case $f(\Omega)$ has vertices at the images of the points P and V where the boundary of Ω intersects $x = 0$. We now state Propositions 3.4 and 3.5. Proofs are provided in Appendix A.

Proposition 3.4 Let $\xi \in \Phi_{\text{trap}}$. Then $f(\Omega) \subset \Omega$.

Proposition 3.5 Let $\xi \in \Phi_{\text{trap}}$. Given $\varepsilon > 0$ define

$$\begin{aligned} C_\varepsilon &= C - (\varepsilon, 0), \\ Q_\varepsilon &= Q + \varepsilon^2(C - R), \end{aligned}$$

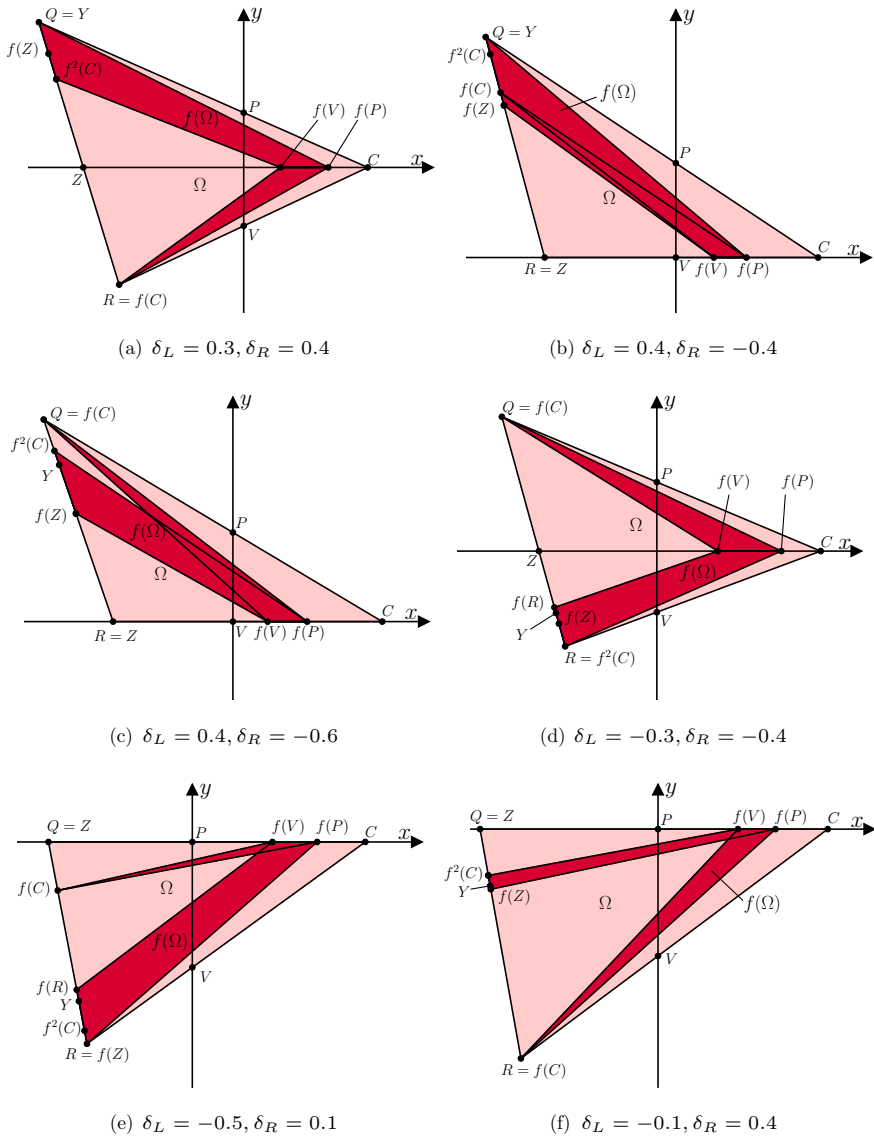


Fig. 5 The triangle Ω and its image under f with $\tau_L = 1.6$, $\tau_R = -1.5$, and six different values of δ_L and δ_R . These correspond to parts A–F of Fig. 3b for different cases for the two vertices of Ω in the left half-plane, denoted Q and R (the third vertex is always C). In each case $f(\Omega) \subset \Omega$ by Proposition 3.4 because the parameter combinations belong to Ω_{trap}

$$R_\varepsilon = R + \varepsilon^2(C - Q),$$

and let Ω_ε be the compact filled triangle with vertices C_ε , Q_ε , and R_ε . Then $f(\Omega_\varepsilon) \subset \text{int}(\Omega_\varepsilon)$, for all sufficiently small $\varepsilon > 0$.

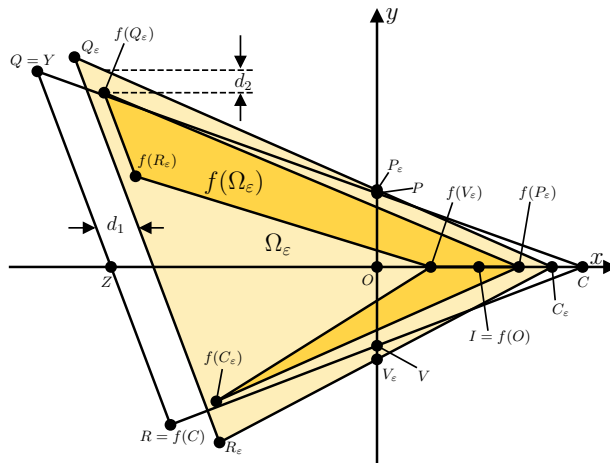


Fig. 6 The trapping region Ω_ε and its image under f . This figure was produced using $\xi = \{1.6, 0.3, -1.5, 0.4\}$, as in Fig. 5A, and $\varepsilon = 0.3$ which is small enough that $f(\Omega_\varepsilon) \subset \text{int}(\Omega_\varepsilon)$ in accordance with Proposition 3.5

The triangle Ω_ε is shown in Fig. 6 for one combination of parameter values. It has been defined so that its left edge lies to the right of $W_0^s(Y)$ and is parallel to $W_0^s(Y)$. Due to the saddle nature of Y , this edge shifts further to the right when iterated under f . Also the left edge is an order ε^2 distance from $W_0^s(Y)$ to ensure $f(C_\varepsilon)$ lies to the right of the image of this edge.

4 Invariant Expanding Cones

In this section, we first define cones and what it means for them to be invariant and expanding under an arbitrary matrix A . We then focus on the Jacobian matrices A_L and A_R of the normal form (1), because in order for a cone to establish chaos in (1) it needs to be invariant and expanding for both A_L and A_R . We then explicitly construct such a cone for any $\xi \in \Phi_{\text{cone}}$ (Proposition 3), and use this to prove Theorem 2.2.

Definition 4.1 A set $C \subset \mathbb{R}^2$ is a *cone* if $\alpha v \in C$ for all $v \in C$ and $\alpha \in \mathbb{R}$.

Definition 4.2 Given $A \in \mathbb{R}^{2 \times 2}$, a cone $C \subset \mathbb{R}^2$ is

- (i) *invariant* under A if $Av \in C$ for all $v \in C$, and
- (ii) *expanding* under A if there exists $c > 1$ such that $\|Av\| \geq c\|v\|$ for all $v \in C$.

In this paper we use the Euclidean norm $\|v\| = \sqrt{v_1^2 + v_2^2}$ and it suffices to consider cones of the form

$$\Psi_K = \left\{ \alpha \begin{bmatrix} 1 \\ m \end{bmatrix} \mid \alpha \in \mathbb{R}, m \in K \right\}, \quad (25)$$

where K is an interval. Since $v \mapsto Av$ is a linear map, to verify invariance and expansion of a cone Ψ_K , it suffices to verify properties (i) and (ii) for vectors of the form $v = \begin{bmatrix} 1 \\ m \end{bmatrix}$:

Lemma 4.3 *If $Av \in \Psi_K$ for all $v = \begin{bmatrix} 1 \\ m \end{bmatrix}$ with $m \in K$, then Ψ_K is invariant under A .*

If there exists $c > 1$ such that $\|Av\| \geq c\|v\|$ for all $v = \begin{bmatrix} 1 \\ m \end{bmatrix}$ with $m \in K$, then Ψ_K is expanding under A .

Now we focus on the Jacobian matrices

$$A_J = \begin{bmatrix} \tau_J & 1 \\ -\delta_J & 0 \end{bmatrix},$$

where $J \in \{L, R\}$, of the normal form (1). The slope of $v = \begin{bmatrix} 1 \\ m \end{bmatrix}$ is m and the slope of $A_J v$ is

$$G_J(m) = -\frac{\delta_J}{\tau_J + m}, \quad (26)$$

assuming $m \neq -\tau_J$. Notice

$$\frac{dG_J(m)}{dm} = \frac{\delta_J}{(\tau_J + m)^2}, \quad (27)$$

thus $G_J(m)$ is increasing if $\delta_J > 0$, decreasing if $\delta_J < 0$, and flat if $\delta_J = 0$. In any case, $G_J(m)$ is monotone and so in order to verify invariance under A_J it suffices to consider the endpoints of K :

Lemma 4.4 *Let $\tau_J, \delta_J \in \mathbb{R}$ and $K = [a, b]$ be an interval with $-\tau_J \notin K$. If $a \leq G_J(a) \leq b$ and $a \leq G_J(b) \leq b$ then Ψ_K is invariant under A_J .*

Proof Since $-\tau_J \notin K$, by (26) and (27) $G_J(m)$ is continuous and monotone on K . Thus for any $m \in K$, $G_J(m)$ is equal to or lies between the values $G_J(a)$ and $G_J(b)$. Thus $a \leq G_J(m) \leq b$. That is, the slope of $A_J \begin{bmatrix} 1 \\ m \end{bmatrix}$ belongs to K , thus $A_J \begin{bmatrix} 1 \\ m \end{bmatrix} \in \Psi_K$. Hence Ψ_K is invariant under A_J by Lemma 4.3. $\square$

Next we introduce the function

$$H_J(m) = \left\| A_J \begin{bmatrix} 1 \\ m \end{bmatrix} \right\|^2 - \left\| \begin{bmatrix} 1 \\ m \end{bmatrix} \right\|^2 = \tau_J^2 + \delta_J^2 - 1 + 2\tau_J m. \quad (28)$$

It is easy to show that if $H_J(m) > 0$ for all m in a compact interval K , then Ψ_K is expanding under A_J . Since $H_J(m)$ is a linear function of m it again suffices to consider the endpoints of K :

Lemma 4.5 Let $\tau_J, \delta_J \in \mathbb{R}$ and $K = [a, b]$ be an interval. If $H_J(a) > 0$ and $H_J(b) > 0$ then Ψ_K is expanding under A_J .

Proof Let $h = \min[H_J(a), H_J(b)] > 0$. By (28), $H_J(m) \geq h$ for all $m \in K$. Then for any $m \in K$ the vector $v = \begin{bmatrix} 1 \\ m \end{bmatrix}$ satisfies

$$\|A_J v\|^2 = H_J(m) + \|v\|^2 \geq h + \|v\|^2 = \left(\frac{h}{\|v\|^2} + 1\right) \|v\|^2 \geq \left(\frac{h}{n} + 1\right) \|v\|^2,$$

where $n = \max_{m \in K} (1 + m^2)$. Thus Ψ_K is expanding under A_J (with expansion factor $c = \sqrt{\frac{h}{n} + 1} > 1$) by Lemma 4.3. $\square$

To prove chaos in (1) we need to choose $K = [a, b]$ so that Ψ_K is invariant under A_L and A_R . This favors the interval K being relatively large. However, we want K to be as small as possible in order to maximize the parameter region over which it is expanding under A_L and A_R . This balancing act motivates the following calculations that form the basis of our definition of K given below in Proposition 4.6.

For each $J \in \{L, R\}$, the fixed point equation $G_J(m) = m$ is quadratic in m . If $\delta_J \neq 0$ and $\delta_J < \frac{\tau_J^2}{4}$, then G_J has exactly two fixed points

$$q_J = -\frac{\tau_J}{2} \left(1 - \sqrt{1 - \frac{4\delta_J}{\tau_J^2}}\right), \quad r_J = -\frac{\tau_J}{2} \left(1 + \sqrt{1 - \frac{4\delta_J}{\tau_J^2}}\right). \quad (29)$$

In order for Ψ_K to be invariant under A_L and A_R , we define K so that it contains q_L and q_R , see Fig. 7. So the smallest interval we consider is $K = [q_L, q_R]$. In the orientation-preserving case ($\delta_L, \delta_R > 0$), this interval indeed gives invariance, as shown in Glendinning and Simpson (2021). In the orientation-reversing case ($\delta_L, \delta_R < 0$), invariance requires that K contains a period-two solution. The equation $(G_R \circ G_L)(m) = m$ is quadratic in m with discriminant

$$\theta_1(\xi) = (\delta_L + \delta_R - \tau_L \tau_R)^2 - 4\delta_L \delta_R,$$

repeating (18). So if $\theta_1(\xi) > 0$ there are two period-two solutions, Fig. 7-(IV). Of these the inner-most solution is $\{\tilde{a}, \tilde{b}\}$, where

$$\tilde{a} = \frac{\delta_L - \delta_R - \tau_L \tau_R - \sqrt{\theta_1(\xi)}}{2\tau_R}, \quad \tilde{b} = \frac{\delta_R - \delta_L - \tau_L \tau_R - \sqrt{\theta_1(\xi)}}{2\tau_L}, \quad (30)$$

satisfying $G_L(\tilde{a}) = \tilde{b}$ and $G_R(\tilde{b}) = \tilde{a}$. So in this case the smallest interval we can take is $K = [\tilde{a}, \tilde{b}]$, as used by Misiurewicz (1980).

In the non-invertible cases the slope maps G_L and G_R are either both non-negative or both non-positive, see again Fig. 7. Thus a simple and effective choice for one

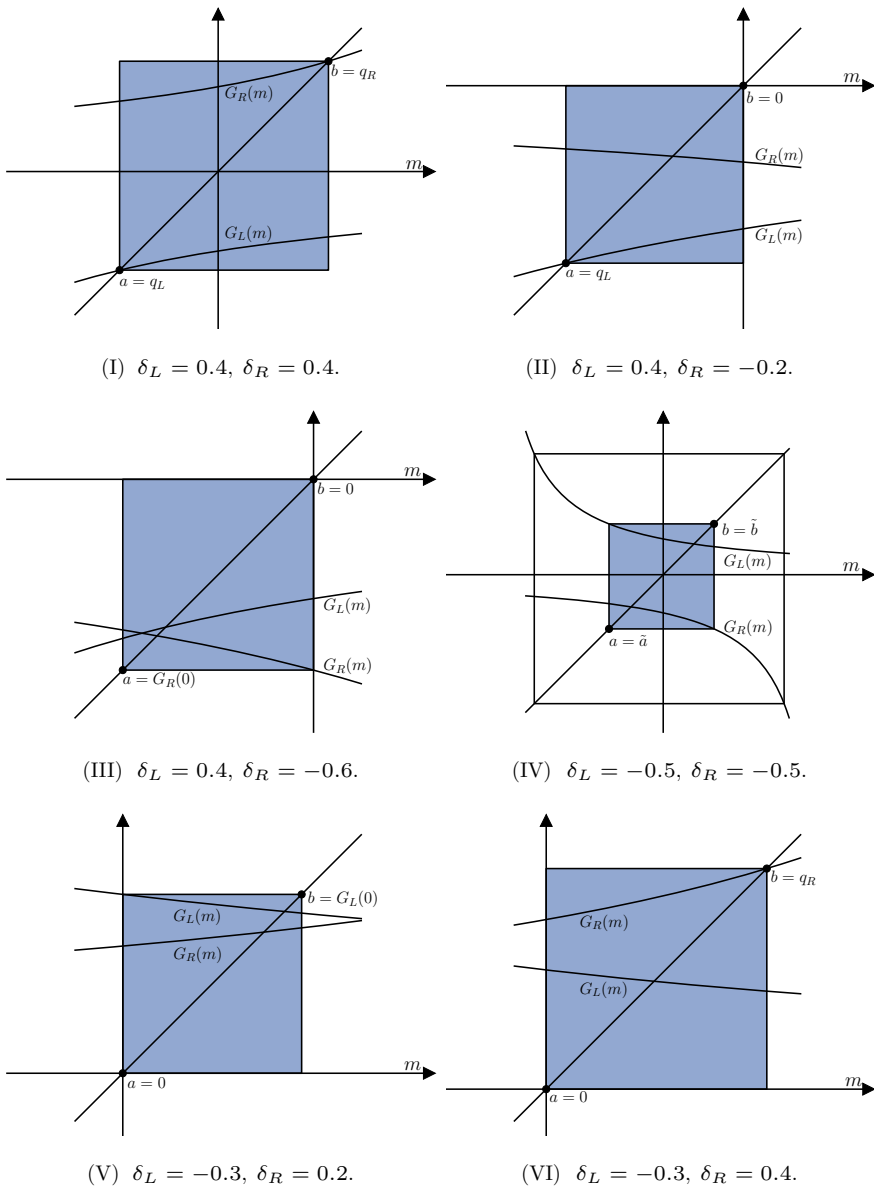


Fig. 7 Cobweb diagrams of the slope maps $G_L(m)$ and $G_R(m)$ (26) with $\tau_L = 1.6$, $\tau_R = -1.5$, and six different values of δ_L and δ_R . These correspond to parts (I)–(VI) of Fig. 4b. In each case $K = [a, b]$, defined by (34) and (35), is forward invariant under G_L and G_R as shown. This implies Ψ_K is invariant under A_L and A_R (see Lemma 4.4). In each case K contains the fixed points q_L and q_R ; in case (IV) K is defined using one of the two period-two solutions

endpoint of K is $m = 0$. In this case the smallest interval leading to invariance uses also one of q_L , q_R , or the image of $m = 0$ under G_L or G_R :

$$G_L(0) = -\frac{\delta_L}{\tau_L}, \quad G_R(0) = -\frac{\delta_R}{\tau_R}. \quad (31)$$

Proposition 4.6 shows that all cases can be accommodated by simply defining a and b as the minimum and maximum of all points suggested above. Recall Φ_{cone} is defined in Sect. 2 as the set of all $\xi \in \Phi$ for which $\theta_1(\xi)$, $\theta_2(\xi)$, and $\theta_3(\xi)$ are positive. The condition $\theta_1(\xi) > 0$ ensures $\tilde{a}$ and $\tilde{b}$ are well-defined, while, if a and b are given by (34) and (35),

$$\theta_2(\xi) = \tau_L^2 + \delta_L^2 - 1 + 2\tau_L a = H_L(a), \quad (32)$$

$$\theta_3(\xi) = \tau_R^2 + \delta_R^2 - 1 + 2\tau_R b = H_R(b), \quad (33)$$

and $\theta_2(\xi) > 0$ and $\theta_3(\xi) > 0$ ensure Ψ_K is invariant and expanding.

Proposition 4.6 *Let $\xi \in \Phi_{\text{cone}}$ and $K = [a, b]$ where*

$$a = \min\left[0, -\frac{\delta_R}{\tau_R}, q_L, \tilde{a}\right], \quad (34)$$

$$b = \max\left[0, -\frac{\delta_L}{\tau_L}, q_R, \tilde{b}\right]. \quad (35)$$

Then Ψ_K is invariant and expanding under A_L and A_R .

Proposition 4.6 is proved in Appendix B via a sequence of direct calculations. We now use Propositions 3.5 and 4.6 to prove the main result.

Proof of Theorem 2.2 Choose any $\xi \in \Phi_{\text{trap}} \cap \Phi_{\text{cone}}$. By Proposition 3.5 there exists a trapping region Ω_ε for f . Then $\bigcap_{n \geq 0} f^n(\Omega_\varepsilon)$ is an attracting set and contains a topological attractor Λ . By Proposition 4.6 there exists a cone Ψ_K that is invariant and expanding for both A_L and A_R with some expansion factor $c > 1$, and Ψ_K contains a non-zero vector.

Choose any $z \in \Lambda$ and let $v \in \Psi_K$ be non-zero. The Lyapunov exponent $\lambda(z, v)$ for z in the direction v is the limiting rate of separation of the forward orbits of z and $z + \Delta v$ for arbitrarily small $\Delta > 0$ (Viana 2014). If the forward orbit of z does not intersect the switching manifold then the derivative of the n th iterate of z under f is well-defined for all $n \geq 1$ and

$$\lambda(z, v) = \limsup_{n \rightarrow \infty} \frac{1}{n} \ln(\|Df^n(x)v\|). \quad (36)$$

Observe

$$Df^n(x) = Df\left(f^{n-1}(x)\right) \cdots Df(f(x))Df(x),$$

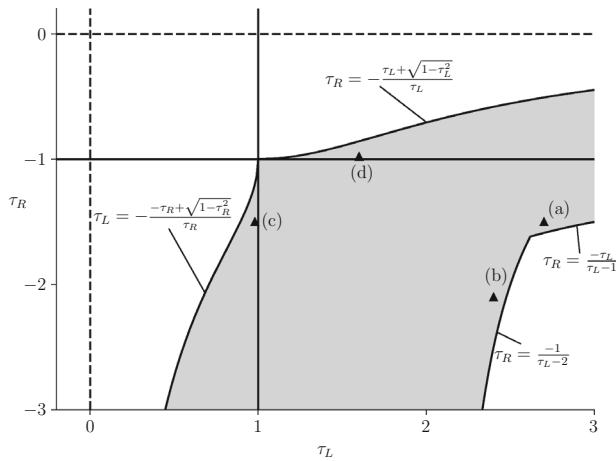


Fig. 8 The shaded region shows where cross-sections of Φ_{trap} and Φ_{cone} , defined by fixing the values of τ_L and τ_R , are both non-empty. As we exit the shaded region through the lower-right boundaries the cross-section of Φ_{trap} vanishes; as we exit through the upper and left boundaries the cross-section of Φ_{cone} vanishes

where each of the n matrices on the right-hand side is either A_L or A_R . By the invariance and expansion of Ψ_K , $\|Df^n(x)v\| \geq c^n \|v\|$ for all n , so $\lambda(z, v) \geq \ln(c) > 0$. If instead the forward orbit of z intersects the switching manifold, $\lambda(z, v)$ can similarly be evaluated and bounded using one-sided directional derivatives because f is piecewise-linear, see Simpson (2022) for details. $\square$

5 Further Remarks on the Parameter Regions Φ_{trap} and Φ_{cone}

In Sect. 2, we described two-dimensional cross-sections of Φ_{trap} and Φ_{cone} defined by fixing the values of $\tau_L > 0$ and $\tau_R < 0$. For the most part larger values of τ_L and $|\tau_R|$ yield smaller cross-sections of Φ_{trap} and larger cross-sections of Φ_{cone} , see Figs. 3 and 4. This is because with larger values of τ_L and $|\tau_R|$ the map is more strongly expanding, hence less amenable for the existence of a trapping region but more amenable for the existence of an invariant expanding cone.

Figure 8 shows critical curves in the (τ_L, τ_R) -plane where the cross-sections vanish entirely. To explain this figure we treat the critical curves one by one. First consider (τ_L, τ_R) at a point just above the critical curve $\tau_R = \frac{-\tau_L}{\tau_L - 1}$. Here the Φ_{trap} cross-section has three vertices, $P^{(1)}$, $P^{(2)}$, and $P^{(3)}$, as shown in Fig. 9a. It is a simple exercise to show that as parameters are varied each vertex reaches the origin $(\delta_L, \delta_R) = (0, 0)$ on the critical curve. For instance the upper vertex is where $\phi_3(\xi) = 0$ and $\phi_4(\xi) = 0$ intersect at $P^{(1)} = \left(0, \frac{(\tau_R - (\tau_R + 1)\tau_L)\tau_L}{1 - \tau_L}\right)$ and solving $P_2^{(1)} = 0$ gives $\tau_R = \frac{-\tau_L}{\tau_L - 1}$. Thus, the Φ_{trap} cross-section contracts to a point and vanishes.

With instead (τ_L, τ_R) at a point just to the left of $\tau_R = \frac{-1}{\tau_L - 2}$, the Φ_{trap} cross-section has three vertices at different points $P^{(4)}$, $P^{(5)}$, and $P^{(6)}$, as shown in Fig. 9b. Explicit

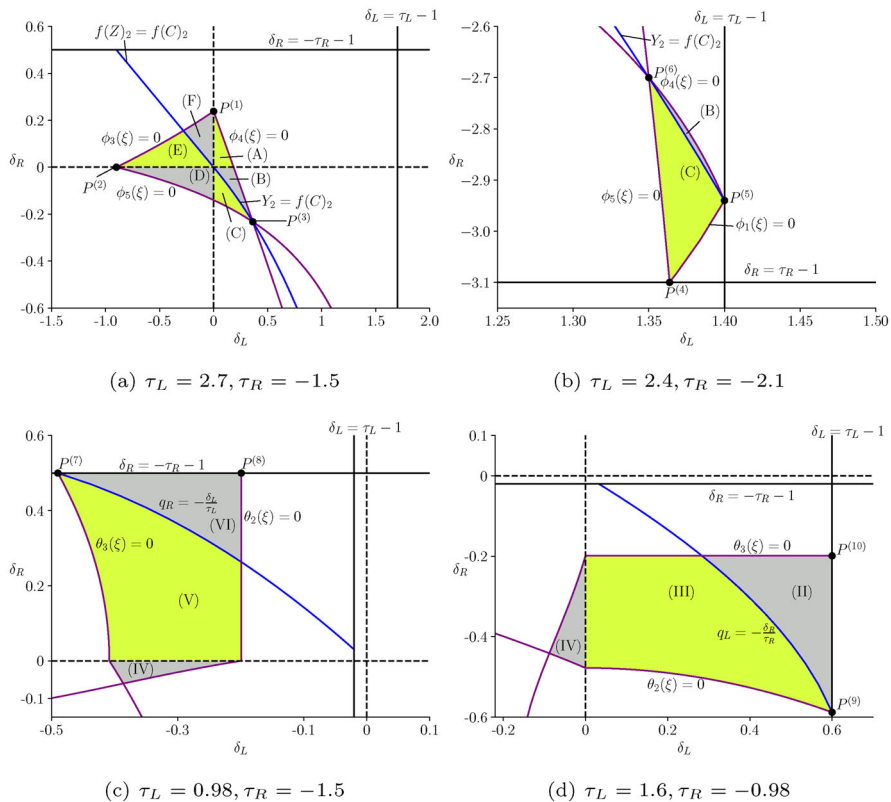


Fig. 9 Further sample cross-sections of Φ_{trap} (panels **a** and **b**) and Φ_{cone} (panels **c** and **d**). These use the parameter points (τ_L, τ_R) shown in Fig. 8

calculations reveal that each vertex reaches the corner $(\delta_L, \delta_R) = (\tau_L - 1, \tau_R - 1)$ when $\tau_R = \frac{-1}{\tau_L - 2}$. For instance, $P^{(4)} = \left(\frac{(\tau_L \tau_R - \tau_R + 1)(\tau_R - 1)}{\tau_R^2}, \tau_R - 1 \right)$, and solving $P_1^{(4)} = \tau_L + 1$ gives $\tau_R = \frac{-1}{\tau_L - 2}$. Thus here the Φ_{trap} cross section again vanishes.

With τ_L just less than 1 and $\tau_R < -1$ the Φ_{cone} cross-section appears as in Fig. 9c. As parameters are varied the Φ_{cone} cross-section vanishes when the vertices $P^{(7)} = (\tau_L(\tau_R + 1), -(\tau_R + 1))$ and $P^{(8)} = (-\sqrt{1 - \tau_L^2}, -(\tau_R + 1))$ coincide. Solving $P_1^{(7)} = P_1^{(8)}$ yields the critical curve $\tau_R = -\frac{\tau_L + \sqrt{1 - \tau_L^2}}{\tau_L}$ of Fig. 8. Similarly with $\tau_L > 1$ and τ_R just greater than -1 the Φ_{cone} cross-section appears as in Fig. 9d. The cross-section vanishes when $P^{(9)}$ and $P^{(10)}$ coincide on $\tau_L = -\frac{-\tau_R + \sqrt{1 - \tau_R^2}}{\tau_R}$.

The geometry and topology of cross-sections of $\Phi_{\text{trap}} \cap \Phi_{\text{cone}}$ admit many possibilities and a complete analysis is beyond the scope of this paper. Here we examine one example, Fig. 10. Here the cross-section of $\Phi_{\text{trap}} \cap \Phi_{\text{cone}}$ is bounded by the following curves (going anticlockwise): $\delta_L = \tau_L - 1$, $\delta_R = -\tau_R - 1$, $\phi_4(\xi) = 0$, $\phi_3(\xi) = 0$,

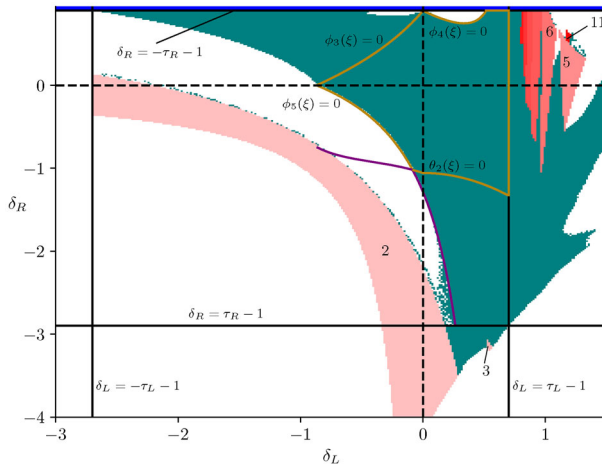


Fig. 10 A two-parameter bifurcation diagram of (1) with $\tau_L = 1.7$ and $\tau_R = -1.9$. The boundary of $\Phi_{\text{trap}} \cap \Phi_{\text{cone}}$ is shown in yellow. The colored regions show the result of a numerical simulation (white: no attractor; green: chaotic attractor; other colors: periodic attractor). Some periods are indicated (e.g., in the large pink region the map has a stable period-2 solution). In the blue strip at the top the fixed point X is stable (color figure online)

$\phi_5(\xi) = 0$, and $\theta_2(\xi) = 0$ (which has a kink at $\delta_L = 0$). The first two of these curves are boundaries of our overall parameter region Φ , the next three curves are boundaries of Φ_{trap} , and the last curve is a boundary of Φ_{cone} .

Figure 10 also shows the result of a simple numerical simulation to investigate the nature of the attractor. For each point in a 300×300 equispaced grid of (δ_L, δ_R) values, we computed 10^7 iterates of the forward orbit using a random initial condition. Green points are where an estimate of the maximal Lyapunov exponent was positive, white points are where the orbit appeared to diverge (its norm exceeded 10^4), and other points are where there exists a stable periodic solution (determined by solving for periodic solutions exactly).

Note the numerical simulation gives an imperfect picture. For example, we believe no attractor exists immediately to the left of the heteroclinic bifurcation $\phi_5(\xi) = 0$, yet some green points are present here (with small $\delta_L > 0$ and $-3 < \delta_R < -2$) because orbits often experience a long transient before diverging and 10^7 iterations are insufficient to detect this.

Nevertheless, the numerics effectively highlights the fact that three of the boundaries of $\Phi_{\text{trap}} \cap \Phi_{\text{cone}}$ are bifurcations where the chaotic attractor is destroyed, so these boundaries cannot be improved upon. As we cross $\delta_R = -\tau_R - 1$ the fixed point X becomes stable, $\phi_4(\xi) = 0$ is a homoclinic bifurcation where the attractor is destroyed (Banerjee et al. 1998), and $\phi_5(\xi) = 0$ is a heteroclinic bifurcation where the attractor is destroyed. Elsewhere the attractor is destroyed in a variety of other global bifurcations. Above $\phi_3(\xi) = 0$ and to the right of $\delta_L = \tau_L - 1$ the trapping region construction of Sect. 3 fails. An alternate construction that partially deals with this is described in Simpson (2022). Below $\theta_2(\xi) = 0$ the cone Ψ_K of Sect. 4 fails to be expanding. For some parameter combinations below $\theta_2(\xi) = 0$ it is possible to construct an invariant

expanding cone, and hence verify the presence of chaos, by using an induced map (Glendinning and Simpson 2022).

6 Discussion

We have formally demonstrated robust chaos in the two-dimensional border-collision normal form with parameter combinations for which the map is orientation-reversing and non-invertible. Specifically we identified an open parameter region Φ_{trap} , where the map has a trapping region, and an open parameter region Φ_{cone} , where it has an invariant expanding cone, see Figs. 3 and 4. Throughout $\Phi_{\text{trap}} \cap \Phi_{\text{cone}}$ the map has an attractor with a positive Lyapunov exponent, Theorem 2.2.

In Fig. 10, we considered a typical two-dimensional cross-section of parameter space. Numerical results showed robust chaos throughout $\Phi_{\text{trap}} \cap \Phi_{\text{cone}}$, corroborating Theorem 2.2 as expected. The robust chaos terminates at three boundaries of $\Phi_{\text{trap}} \cap \Phi_{\text{cone}}$: $\delta_R = -\tau_R - 1$, $\phi_4(\xi) = 0$, and $\phi_5(\xi) = 0$, and appears to persist beyond the other boundaries. We expect our construction could be adapted to verify robust chaos beyond these boundaries, and already this has been achieved in some cases (Glendinning and Simpson 2022; Simpson 2022).

Notice in Fig. 10 the cross-section of $\Phi_{\text{trap}} \cap \Phi_{\text{cone}}$ includes a neighborhood of $(\delta_L, \delta_R) = (0, 0)$. This is the case for many values of τ_L and τ_R and is a significant achievement of this paper because it shows robust chaos is not lost as we cross from the orientation-preserving setting to the orientation-reversing and non-invertible settings. Thus the presence of robust chaos is not dependent on the global topological properties of the map. Moreover, this provides a path for robust chaos to be demonstrated in higher dimensional maps. The n -dimensional border-collision normal form Simpson (2016) can have two saddle fixed points: X with an eigenvalue $\lambda_R^u < -1$, and Y with an eigenvalue $\lambda_L^u > 1$. If all other eigenvalues associated with X and Y are sufficiently small in absolute value (which in two dimensions means (δ_L, δ_R) is sufficiently close to $(0, 0)$), and with appropriate constraints on the values of λ_R^u and λ_L^u , we believe the map must have a chaotic attractor, and this is a tantalizing avenue for future research.

A Proofs of Propositions 3.4 and 3.5

For clarity, we break each of the proofs into steps.

Proof of Proposition 3.4. Step 1 — Characterize $f(\Omega)$.

The vertices Q and R lie in the left half-plane, while C lies in the right half-plane. Let P denote the intersection of $\overline{QC}$ (the line segment from Q to C) with $x = 0$, and V denote the intersection of $\overline{RC}$ with $x = 0$, see Fig. 5. From the formula (9) for C , the y -components of these points are given in terms of Q and R by

$$P_2 = \frac{\lambda_L^{u^3} Q_2}{\lambda_L^{u^3} + [\lambda_L^u - Q_2(1 - \lambda_L^u)]\phi_1(\xi)}, \quad (37)$$

$$V_2 = \frac{\lambda_L^u{}^3 R_2}{\lambda_L^u{}^3 + [\lambda_L^u - R_2(1 - \lambda_L^u)]\phi_1(\xi)}. \quad (38)$$

So Ω is the union of the quadrilateral Ω_L in the left half-plane with vertices P , Q , R , and V , and the triangle Ω_R in the right half-plane with vertices C , P , and V . Thus $f(\Omega) = f(\Omega_L) \cup f(\Omega_R)$, where $f(\Omega_L)$ and $f(\Omega_R)$ are polygons because each piece of f is affine. Thus since Ω is convex, to prove $f(\Omega) \subset \Omega$ it suffices to show that the vertices of $f(\Omega_L)$ and $f(\Omega_R)$ belong to Ω . These vertices are the points $f(C)$, $f(P)$, $f(Q)$, $f(R)$, and $f(V)$.

Step 2 — Show C , Q , and R map to Ω .

Certainly $f(C) \in \Omega$ by the definition of Ω . We now show $f(Q), f(R) \in \overline{QR}$ (the left edge of Ω). If $\delta_L > 0$ then $\lambda_L^s > 0$, so $f(Q) \in \overline{QY}$ and $f(R) \in \overline{RY}$ so certainly $f(Q), f(R) \in \overline{QR}$. Also if $\delta_L = 0$, then $f(Q) = f(R) = Y = Z \in \overline{QR}$. Finally if $\delta_L < 0$, then $f(Q)$ and $f(R)$ lie below $y = 0$, so lie below Z , and hence below Q . In this case Q is either $f(C)$ or Z (because $Y, f(Z)$, and $f^2(C)$ lie below $y = 0$), thus $f(Q)$ lies on or above R , by the definition of R . Also $\lambda_L^s < 0$, thus $f(R)$ lies above $f(Q)$, and hence above R . Thus in any case $f(Q), f(R) \in \overline{QR}$.

Step 3 — Show P and V map to Ω .

The points $f(P)$ and $f(V)$ lie on $y = 0$, specifically

$$f(P) = \begin{bmatrix} P_2 + 1 \\ 0 \end{bmatrix}, \quad f(V) = \begin{bmatrix} V_2 + 1 \\ 0 \end{bmatrix}, \quad (39)$$

where P_2 and V_2 are given by (37) and (38). Also V lies above S , thus $f(V)$ lies to the right of $Z = f(S)$. Hence it remains for us to show that $f(P)$ lies at or to the left of C , that is $C_1 - (P_2 + 1) \geq 0$. To do this we consider the various possibilities for Q in turn. There are three cases: Q is either Y , Z , or $f(C)$. This is because $f(Z)$ cannot lie above Y , while $f^2(C)$ cannot lie above Y if $\delta_L > 0$ and cannot lie above Z if $\delta_L \leq 0$.

Case 1: With $Q = Z$ we have $P_2 = 0$. Thus $C_1 - (P_2 + 1) > 0 = C_1 - 1 > 0$ because $\phi_3(\xi) > 0$.

Case 2: With $Q = Y$, by substituting Y_2 , given by (4), in place of Q_2 in (37) we obtain

$$C_1 - (P_2 + 1) = \left(1 + \frac{\lambda_L^s \lambda_L^u{}^2}{\lambda_L^u{}^2(1 - \lambda_L^s) + \phi_1(\xi)} \right) \frac{\phi_4(\xi)}{\phi_1(\xi)(\lambda_L^u - 1)}.$$

This case requires $\delta_L \geq 0$, thus $\lambda_L^s \geq 0$. Also $C_1 - (P_2 + 1) > 0$ because $\phi_1(\xi) > 0$ and $\phi_4(\xi) > 0$.

Case 3: With $Q = f(C)$ we similarly obtain

$$C_1 - (P_2 + 1) = \left(1 + \frac{\lambda_R^s \lambda_L^u{}^2}{\lambda_L^u(\lambda_L^u - \delta_R)\phi_1(\xi)} \right) \frac{\phi_5(\xi)}{\phi_1(\xi)(\lambda_L^u - 1)}.$$

This case requires $\delta_R \leq 0$ (so that $f(C)_2 \geq 0$), thus $\lambda_R^s \geq 0$. Also $\phi_1(\xi) > 0$ and $\phi_5(\xi) > 0$, thus $C_1 - (P_2 + 1) > 0$. $\square$

Proof of Proposition 3.5. Step 1 — Characterize $f(\Omega_\varepsilon)$.

Assume $\varepsilon > 0$ is sufficiently small that Q_ε and R_ε lie to the left of $x = 0$ and C_ε lies to the right of $x = 0$. Then the line segment $\overline{C_\varepsilon Q_\varepsilon}$ intersects $x = 0$ at a unique point P_ε , as does $\overline{C_\varepsilon R_\varepsilon}$ at a point V_ε . Similar to the previous proof, it remains for us to show that C_ε , P_ε , Q_ε , R_ε , and V_ε map to the interior of Ω_ε .

Step 2 — Show C_ε maps to the interior of Ω_ε .

Let $O = (0, 0)$ denote the origin and $I = (1, 0)$ be its image under f . Also let $\ell = \overline{CO}$. This line segment maps under the right piece of f to the line segment from $f(C_\varepsilon)$ to I . Since $C_\varepsilon \in \ell$ is an order ε distance from C , its image $f(C_\varepsilon) \in f(\ell)$ is an order ε distance from $f(C)$, which belongs to $\overline{QR}$. Since $\overline{Q_\varepsilon R_\varepsilon}$ is an order ε^2 distance from $\overline{QR}$, $f(C_\varepsilon)$ must lie to the right of $\overline{Q_\varepsilon R_\varepsilon}$ for sufficiently small $\varepsilon > 0$.

Also, $f(\ell)$ lies inside the triangle QRI . Since C_ε lies to the right of I (because $\phi_3(\xi) > 0$ and assuming ε is sufficiently small), this triangle lies below $\overline{Q_\varepsilon C_\varepsilon}$ and above $\overline{R_\varepsilon C_\varepsilon}$. Thus $f(C_\varepsilon)$ lies below $\overline{Q_\varepsilon C_\varepsilon}$ and above $\overline{R_\varepsilon C_\varepsilon}$. Thus $f(C_\varepsilon)$ lies inside all three edges of Ω_ε , hence $f(C_\varepsilon) \in \text{int}(\Omega_\varepsilon)$.

Step 3 — Show P_ε and V_ε map to the interior of Ω_ε .

Let A be any point on $\overline{CZ}$ with $A \neq C, Z$. Then there exists sufficiently small $\varepsilon > 0$ such that $A \in \text{int}(\Omega_\varepsilon)$. We know $f(P)$ and $f(V)$ are located on $\overline{CZ} \setminus \{C, Z\}$, because by construction $Z_1 \leq f(V)_1 < f(P)_1 < C_1$, so belong to $\text{int}(\Omega_\varepsilon)$ for sufficiently small $\varepsilon > 0$. The same is true for $f(P_\varepsilon)$ and $f(V_\varepsilon)$ because $P_\varepsilon \rightarrow P$ and $V_\varepsilon \rightarrow V$ as $\varepsilon \rightarrow 0$.

Step 4 — Show Q_ε and R_ε map to the interior of Ω_ε .

For brevity we just show $f(Q_\varepsilon) \in \text{int}(\Omega_\varepsilon)$ ($f(R_\varepsilon) \in \text{int}(\Omega_\varepsilon)$ can be shown similarly). Let $d_1 > 0$ be the distance that $\overline{Q_\varepsilon R_\varepsilon}$ lies to the right of $\overline{QR}$, see Fig. 6. Then $f(Q_\varepsilon)$ lies a distance $\lambda_L^u d_1$ to the right of $\overline{QR}$, as Y is a saddle fixed point with stable direction $\overline{QR}$ and unstable eigenvalue λ_L^u . Since $\lambda_L^u > 1$, the point $f(Q_\varepsilon)$ lies to the right of the line $\overline{Q_\varepsilon R_\varepsilon}$.

If $Q \neq Y$, then $f(Q)$ lies an order 1 distance below $\overline{CQ}$, thus $f(Q_\varepsilon)$ lies below $\overline{C_\varepsilon Q_\varepsilon}$ for sufficiently small $\varepsilon > 0$. Now consider the case $Q = Y$. As shown in Fig. 6 let d_2 be the vertical displacement from $f(Q_\varepsilon)$ upwards to the line $\overline{Q_\varepsilon C_\varepsilon}$ (we will show $d_2 > 0$). By direct calculations $d_2 = \beta\varepsilon^2 + O(\varepsilon^3)$ where

$$\beta = \delta_L(C_1 - R_1)(C_1 + (\tau_L - 2)Y_1) - R_2(C_1 + (\delta_L - 1)Y_1).$$

Let $p = C_1 - D_1$ and notice $p > 0$ by (13) because $\phi_4(\xi) > 0$. Using also $D_1 = (1 - \lambda_L^u)Y_1$ by (4) and (11), we obtain

$$\beta = \delta_L(C_1 - R_1)(p - Y_1(1 - \lambda_L^s)) - R_2(p - Y_1\lambda_L^u(1 - \lambda_L^s)),$$

which is positive by inspection (e.g., $Y_1 < 0$). This shows that $d_2 > 0$ for sufficiently small values of ε , that is $f(Q_\varepsilon)$ lies below the upper edge of Ω_ε . By similar calculations one can show that $f(Q_\varepsilon)$ lies above the lower edge of Ω_ε , and therefore $f(Q_\varepsilon) \in \text{int}(\Omega_\varepsilon)$. $\square$

B Proof of Proposition 4.6

We first prove three lemmas. Lemma B1 provides bounds on the other fixed points r_L and r_R of the slope maps G_L and G_R . Lemma B2 shows that $\tilde{a}$ and $\tilde{b}$ are attained by the minimum and maximum values a and b when the map is orientation-reversing. Lemma B3 provides additional bounds on the values of a and b .

Lemma B1 *Let $\xi \in \Phi$. Then $r_L < \frac{1-\delta_L^2-\tau_L^2}{2\tau_L}$ and $r_R > \frac{1-\delta_R^2-\tau_R^2}{2\tau_R}$.*

Proof We have $\delta_L + 1 < \tau_L$, hence $(\delta_L + 1)^2 < \tau_L^2$, and so $(\delta_L - 1)^2 < \tau_L^2 - 4\delta_L$. By multiplying the last two inequalities together we obtain $(\delta_L^2 - 1)^2 < \tau_L^2(\tau_L^2 - 4\delta_L)$, so $\delta_L^2 - 1 < \tau_L \sqrt{\tau_L^2 - 4\delta_L}$ which rearranges to $r_L < \frac{1-\delta_L^2-\tau_L^2}{2\tau_L}$ using (29). The result for r_R follows similarly. $\square$

Lemma B2 *With the assumptions of Proposition 4.6, $a = \tilde{a}$ if and only if $\delta_L \leq 0$ and $\delta_R \leq 0$; similarly $b = \tilde{b}$ if and only if $\delta_L \leq 0$ and $\delta_R \leq 0$.*

Proof First suppose $\delta_L \leq 0$ and $\delta_R \leq 0$. Then $\tau_L \tau_R \delta_L \geq 0$ (also $\theta_1(\xi) > 0$ by assumption), so we can use (30) to obtain

$$2\tau_R(\tilde{a} + \tau_L) = -\sqrt{\theta_1(\xi)} - \sqrt{\theta_1(\xi) + 4\tau_L \tau_R \delta_L} < 0.$$

Thus $\tilde{a} > -\tau_L$ (because $\tau_R < 0$). Also $\tilde{b} < -\tau_R$ by a similar argument.

Notice $\delta_L \leq 0$ implies $G_L(m) \geq 0$ for all $m > -\tau_L$, so $q_L \geq 0$, $G_L(0) = -\frac{\delta_L}{\tau_L} \geq 0$, and $G_L(\tilde{a}) = \tilde{b} \geq 0$. Similarly $G_R(m) \leq 0$ for all $m < -\tau_R$, so $q_R \leq 0$, $G_R(0) = -\frac{\delta_R}{\tau_R} \leq 0$, and $G_R(\tilde{b}) = \tilde{a} \leq 0$. Also $G_L(m)$ is non-increasing, so $\tilde{a} \leq 0$ implies $G_L(\tilde{a}) \geq G_L(0)$. Thus $\tilde{b} \geq -\frac{\delta_L}{\tau_L} \geq 0 \geq q_R$, so $b = \tilde{b}$. Similarly $\tilde{a} \leq -\frac{\delta_R}{\tau_R} \leq 0 \leq q_L$, so $a = \tilde{a}$.

Conversely suppose $a = \tilde{a}$. Then $\tilde{a} = G_R(\tilde{b}) \leq 0$, so $\delta_R \leq 0$. Thus $G_R(m)$ is non-increasing, so $\tilde{a} = G_R(\tilde{b}) \leq G_R(0)$ implies $\tilde{b} \geq 0$. Thus $\tilde{b} = G_L(\tilde{a}) \geq 0$, so $\delta_L \leq 0$, as required (also $b = \tilde{b}$ implies $\delta_L \leq 0$ and $\delta_R \leq 0$ in a similar fashion). $\square$

Lemma B3 *With the assumptions of Proposition 4.6, $-\tau_L < a$ and $b < -\tau_R$.*

Proof For brevity, we just show $-\tau_L < a$ ($b < -\tau_R$ can be shown similarly). Using $\theta_2(\xi) > 0$ and Lemma B1 we obtain

$$a > \frac{1 - \delta_L^2 - \tau_L^2}{2\tau_L} > r_L. \quad (40)$$

Thus, if $\delta_L \geq 0$ then

$$r_L + \tau_L = \frac{\tau_L}{2} \left(1 - \sqrt{1 - \frac{4\delta_L}{\tau_L^2}} \right) \geq 0,$$

and so $a > -\tau_L$. Now suppose $\delta_L < 0$. If $\delta_R \geq 0$ then $a = 0 > -\tau_L$, while if $\delta_R < 0$ then $a = \tilde{a}$ (by Lemma B2) and $-\tau_L < \tilde{a}$ as shown in the proof of Lemma B2. $\square$

Proof of Proposition 4.6 We first show $G_L(a) \geq a$ and $G_L(b) \geq a$. If $\delta_L \leq 0$ then $G_L(m) \geq 0$ for all $m \in K$ (using Lemma B3), so certainly $G_L(a) \geq a$ and $G_L(b) \geq a$. Now suppose $\delta_L > 0$. In this case, $G_L(m) \geq m$ for all $r_L \leq m \leq q_L$ (i.e., at and between the fixed points of G_L). Observe $r_L < a \leq q_L$ by (40) and the definition of a , thus $G_L(a) \geq a$. Also $G_L(m)$ is increasing thus $G_L(b) > G_L(a) \geq a$.

Next we show $G_L(a) \leq b$ and $G_L(b) \leq b$. If $\delta_L \geq 0$ then we have $G_L(m) \leq 0$ for all $m \in K$ (using Lemma B3), so certainly $G_L(a) \leq b$ and $G_L(b) \leq b$. Now suppose $\delta_L < 0$. If $\delta_R \leq 0$ then Lemma B2 implies $a = \tilde{a}$ and $b = \tilde{b} = G_L(\tilde{a})$, so $b = G_L(a)$, if $\delta_R > 0$ then $a = 0$ and $b \geq G_L(0)$, so $b \geq G_L(a)$. Also $G_L(m)$ is decreasing thus $G_L(b) < G_L(a) \leq b$.

Now from Lemma 4.4, we can conclude that Φ_K is invariant under A_L . Invariance under A_R can be proved in a similar fashion.

Next we prove expansion. By (32), $\theta_2(\xi) > 0$ implies $H_L(a) > 0$. Also

$$H_L(b) = \tau_L^2 + \delta_L^2 - 1 + 2\tau_L b = H_L(a) + 2\tau_L(b - a) \quad (41)$$

is positive because $\tau_L > 0$ and $b \geq a$. Thus Ψ_K is expanding for A_L by Lemma 4.5. By a similar argument Ψ_K is also expanding for A_R . $\square$

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